APPLICATION OF THE WEIBULL DISTRIBUTION TO THE DESCRIPTION OF THE SKEW DISTRIBUTION OF CONCRETE COMPRESSIVE STRENGTH

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Hon. Prof. Tibor Kausay

The extremal distribution is noteworthy because, unlike the normal distribution (Gaussian distribution) and t-distribution (Student distribution) prevalent in engineering, it is generally asymmetric, and therefore is well suited for modeling the actual compressive strength distribution of concrete.

Keywords: Weibull distribution, extremal distribution, density funciton, concrete, compressive strength, skewness

1. INTRODUCTION

Mistéth wrote in his studies (1974, 1977, 2000, 2001) that the distribution function of maximum loads follows the upper Weibull distribution, while that of the minimum strength of loadbearing materials follows the lower third extreme (Weibull) distribution. In accordance with *Mistéth, Ujhelyi* (1978, 1985) wrote that probability distribution of the compressive strength for an average quality level concrete production in the lower concrete class range (below C16/20) or for the highly plastic consistency may be approximated by the log-normal distribution with right skewness (extending to the right with positive skewness), while in the higher compressive strength range (beyond C20/25) or for plastic consistency it may be approximated by the lower extremal distribution with skewness to the left (extending to the left with negative skewness). Compressive strength in the middle strength range (C16/20 – C20/25) may be considered to follow the normal distribution.

2. EXTREME VALUE DISTRIBUTIONS (EVD)

An extreme value is either very small or very large value in a probability distribution. These extreme values are found in the tails of a probability distribution (i.e. the distribution's extremities).

Extreme value analysis (EVA) is a branch of statistics dealing with the extreme deviations from the median of probability distributions. EVD has three types which are named after their most famous researchers and each type can be an upper (maximum) and lower (minimum), as can be seen in Table 1.

Based on experiments of *W. Weibull* (1939), for compressive strength distribution of brittle materials (such as for compressive strength distributions of higher strength concretes) the lower *Weibull* extremal value distribution proved to be right from among the three lower extremal value distribution functions *(Rinne*, 2009).

Table 1: Density function of reduced extreme value distributions (Rinne, 2009)

Type	Upper or maximum extreme value distribution	Lower or minimum extreme value distribution
I.	$g_{\mathrm{f}}(u\,\vert 0;1) = \mathrm{e}^{-u\ -\mathrm{e}^{-u}}$ upper *Gumbel* distribution	$g_{\mathrm{a}}(u\,\vert 0;1) = \mathrm{e}^{u\ -\mathrm{e}^{u}}$ lower *Gumbel* distribution
II.	$f_{\mathrm{Fr,f}}(u\vert 0;1;\mathrm{c}) =$ $= \mathrm{c} \times u^{-\mathrm{c}-1} \times \mathrm{e}^{-u^{-\mathrm{c}}}$ upper *Fréchet* distribution	$f_{\mathrm{Fr,a}}(u\vert 0;1;\mathrm{c}) =$ $= \mathrm{c} \times (-u)^{-\mathrm{c}-1} \times \mathrm{e}^{-(-u)^{-\mathrm{c}}}$ lower *Fréchet* distribution
III.	$w_{\mathrm{f}}(u\vert 0;1;\mathrm{c}) =$ $= \mathrm{c} \times (-u)^{\mathrm{c}-1} \times \mathrm{e}^{-(-u)^{\mathrm{c}}}$ upper *Weibull* distribution	$w_{\mathrm{a}}(u\vert 0;1;\mathrm{c}) =$ $= \mathrm{c} \times u^{\mathrm{c}-1} \times \mathrm{e}^{-u^{\mathrm{c}}}$ lower *Weibull* distribution

3. THE WEIBULL DISTRIBUTION

In probability theory and statistics, the Weibull distribution - named after the Swedish mathematician *Weibull* (1951) - is a continuous probability distribution.

The most general form of Weibull's probability density function has three parameters: γ - shape parameter, μ - location parameter and α - scale parameter. The case where $\mu = 0$ and $\alpha = 1$ is called the standard Weibull distribution. The case where $\mu = 0$ is called the 2-parameter Weibull distribution. In the formulation of the single parameter Weibull distribution the only unknown parameter is the scale parameter, η, i.e. we assume that the shape parameter is known a priori from past experience with identical or similar products.

Note: in the literature different letters are used for the parameters. In this paper the shape parameter is marked with "c", the location parameter with "a" and the scale parameter with "b".

The skewness coefficient is the quotient of the third order moment of the area under the curve of the density function with respect to the center of gravity (μ_3) and the third power of the standard deviation (), *(Palotás*, 1979), (*Hartung* et al., 2009):

$$\gamma_{\text{skewness}} = \frac{\mu_3}{{s_n}^3} = \frac{\left(\frac{1}{n} \times \sum_{i=1}^{n}(x_i - \bar{x})^3\right)}{\sqrt[2]{\left(\frac{1}{n} \times \sum_{i=1}^{n}(x_i - \bar{x})^2\right)}^3}$$

4. DETERMINATION OF WEIBULL DISTRIBUTION PARAMETERS

To evaluate results of the compressive strength tests according to the *Mistéth* theory, values of location parameter „a", of scale parameter „b" (in the literature also λ) and shape parameter „c" (in literature also k) of the *Weibull* distribution must be known.

Our assumptions for their determination are that

- skewness factor of the single-parameter, simplified lower *Weibull* distribution equals the empirical skewness factor obtained from the measurements:

$$\gamma_{\text{skewness,Weibull,low}}(u|0;1;c) = \gamma_{\text{skewness,test}}$$

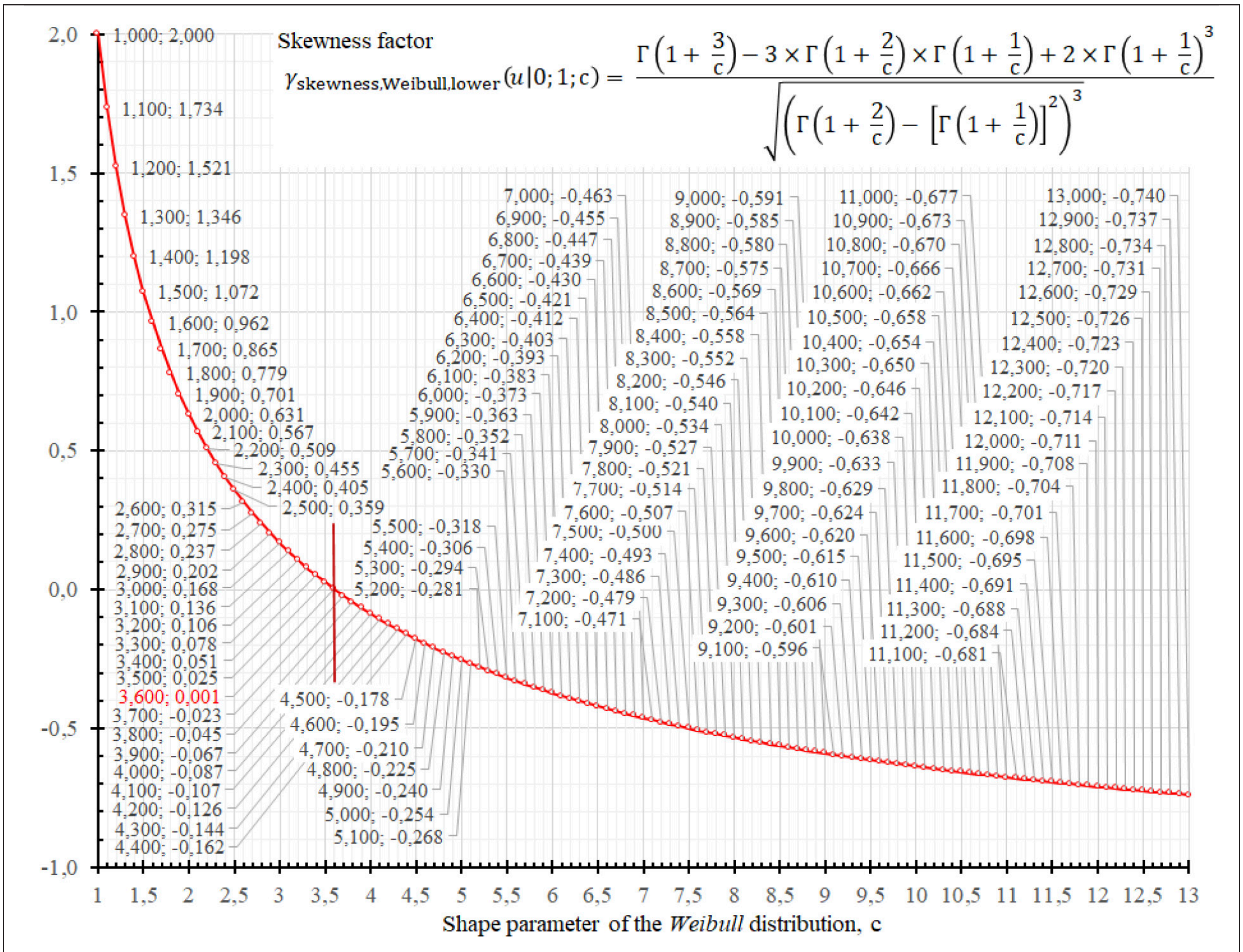
- standard deviation of the two-parameter semi-simplified lower *Weibull* distribution equals to the empirical standard deviation obtained from the measurements:

$$s_{\text{Weibull,low}}(z|0;b;c) = s_{\text{test}}$$

- mean value of the three-parameter lower *Weibull* distribution equals to the empirical average obtained from the measurements:

$$\mu_{\text{Weibull,low}}(x|a;b;c) = f_{\text{cm,test}},$$

Figure 1: Skewness factor of the single-parameter simplified Weibull distribution as a function of the shape parameter „c"



➢ difference between the mean value of the three-parameter lower *Weibull* distribution (which is considered equal with the empirical average) and the mean value of the two-parameter semi-simplified lower *Weibull* distribution equals to the value of the location parameter „a”:

$$a = f_{cm,test} - \mu_{Weibull,low}(z|0;b;c).$$

Recommended steps for the determination of the values of the location parameter „a”, scale parameter „b” and shape parameter „c” are the following:

- Step 1: Into the formula of skewness factor for the single-parameter, simplified lower *Weibull* distribution the empirical skewness factor obtained by the measurements ($\gamma_{skewness,test}$) is introduced and the shape parameter „c” value of the *Weibull* distribution pertinent to the empirical skewness factor is found by implicit iteration:

Figure 2: Standard deviation of the *z* independent variable of the two-parameter lower Weibull distribution semi-simplified assuming that location parameter a = 0, as a function of the scale parameter „b” and the shape parameter „c”, if 1.0 ≤ b ≤ 10.0 and if 1.0 ≤ c ≤ 10.0

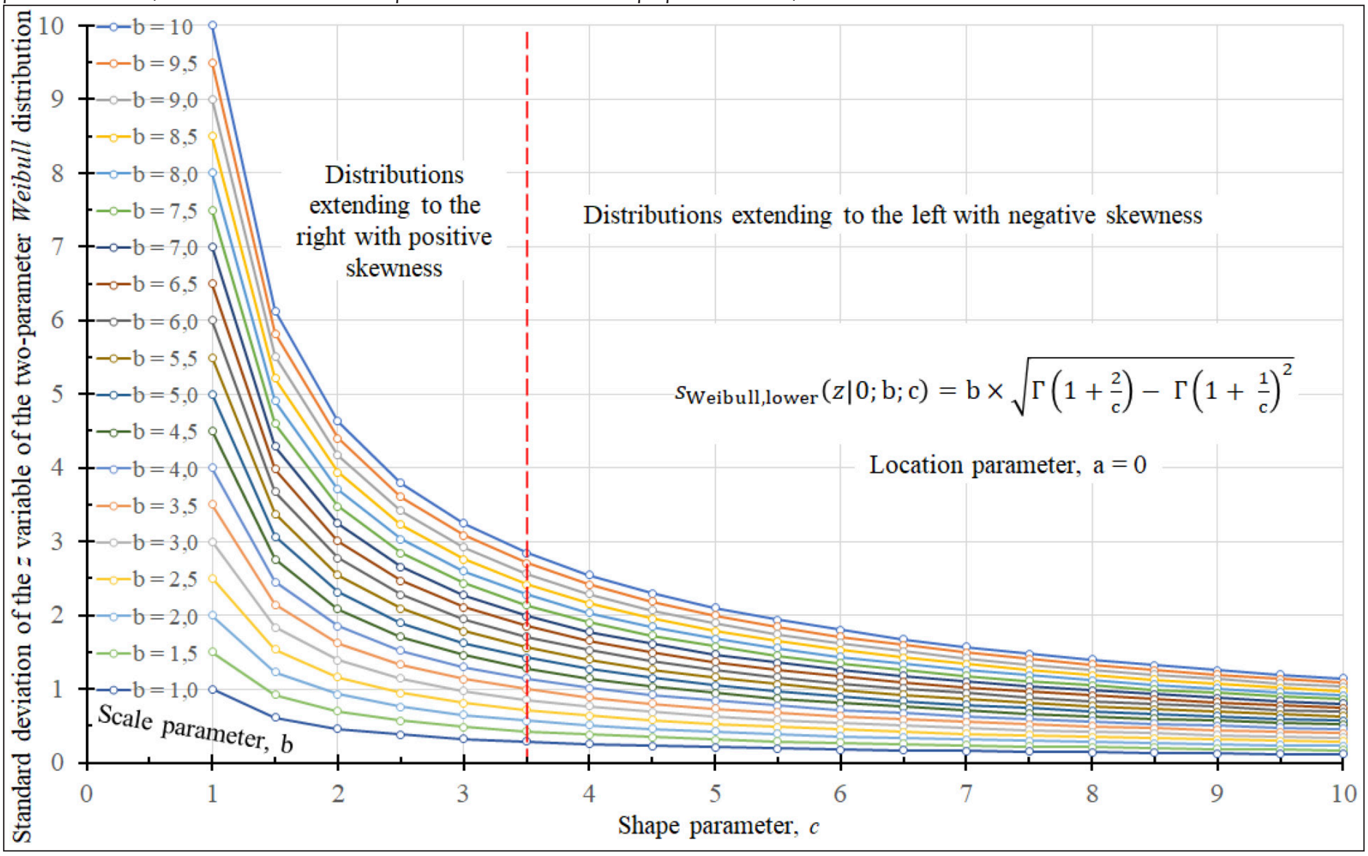


Figure 3: Standard deviation of the z independent variable of the two-parameter lower Weibull distribution semi-simplified by setting the location parameter „a” to 0, as a function of the scale parameter „b” and shape parameter „c”, if 10.0 ≤ b ≤ 50.0 and if 1.0 ≤ c ≤ 10.0

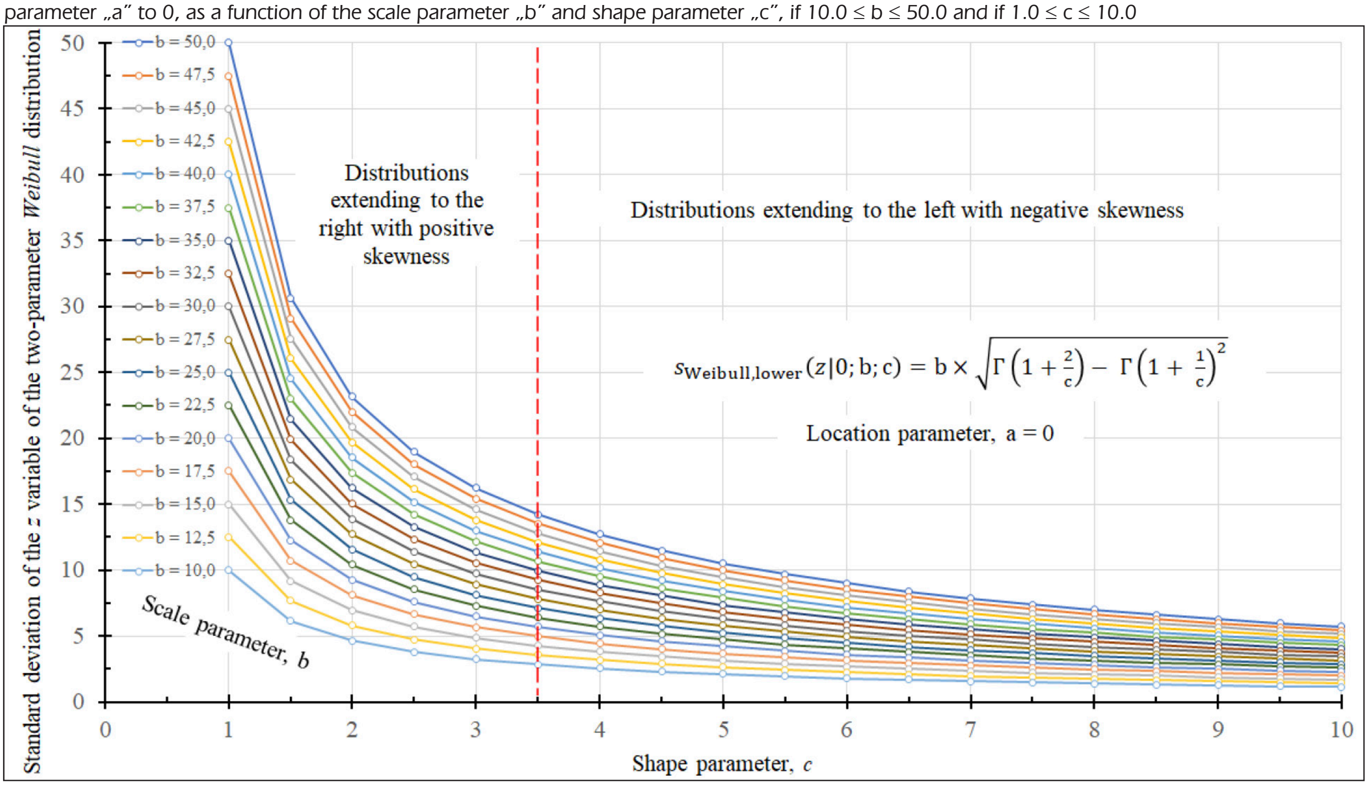


Figure 4: *The quotient of the standard deviation/scale parameter quotient (of the semi-simplified two-parameter lower Weibull distribution (which is proportional to the standard deviation) as function of the shape parameter „c"*

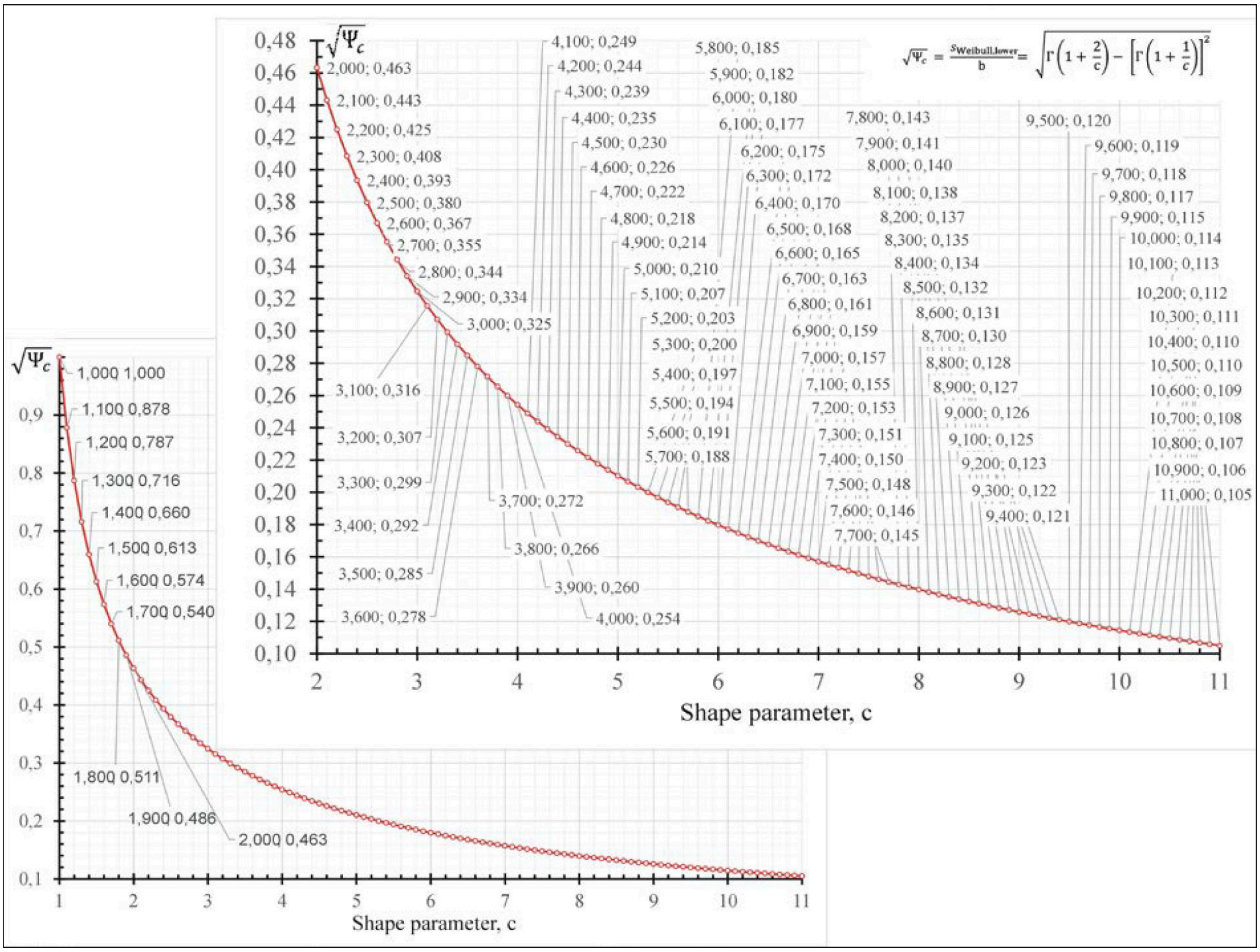
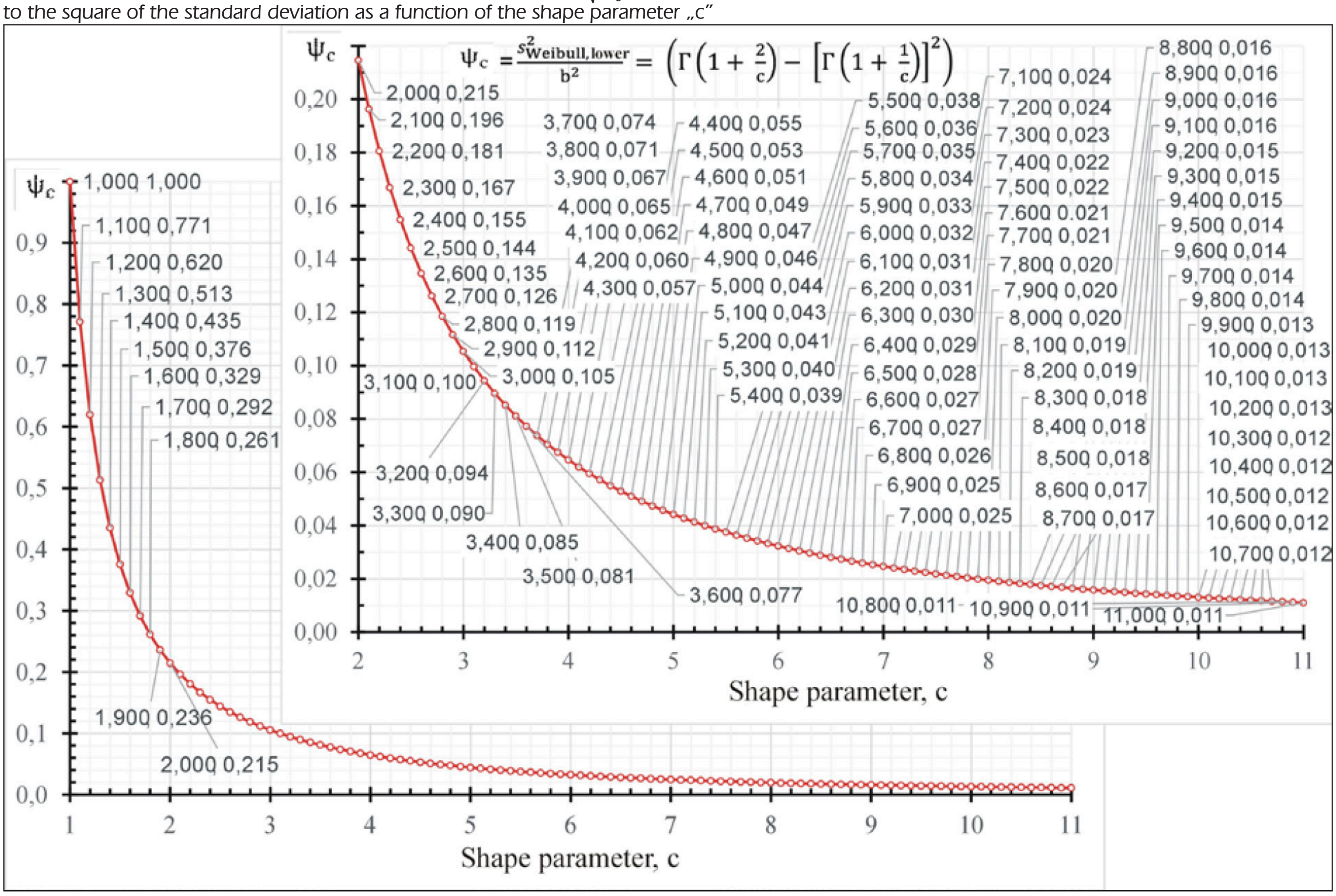


Figure 5: *Squared standard deviation/scale parameter quotient ($\sqrt{\Psi_c}$) of the semi-simplified two-parameter lower Weibull distribution proportional to the square of the standard deviation as a function of the shape parameter „c"*



$$Y_{\text{skewness,Weibull,low}}(u|0;1;c) =$$

$$= \frac{\Gamma\left(1+\frac{3}{c}\right) - 3\times\Gamma\left(1+\frac{2}{c}\right)\times\Gamma\left(1+\frac{1}{c}\right) + 2\times\Gamma\left(1+\frac{1}{c}\right)^{3}}{\sqrt{\left(\Gamma\left(1+\frac{2}{c}\right) - \left[\Gamma\left(1+\frac{1}{c}\right)\right]^{2}\right)^{3}}} = Y_{\text{skewness,test}}$$

Upon the successive approximation (iteration), calculation of skewness factor of the single-parameter lower *Weibull* distribution is repeated until a value approaching the empirical skewness factor best (e.g. to three decimals) is obtained. *Figure 1* may assist this procedure.

- Step 2: Into the formula of the standard deviation of the two-parameter semi-simplified lower *Weibull* distribution the already known value of the shape parameter c and the value of the empirical standard deviation of the concrete

Figure 6: Expected mean value of the *z* independent variable of the two-parameter lower Weibull distribution semi-simplified by the location parameter „a" as a function of the scale parameter „b" and shape parameter „c", if 1.0 ≤ b ≤ 10.0 and if 1.0 ≤ c ≤ 10.0

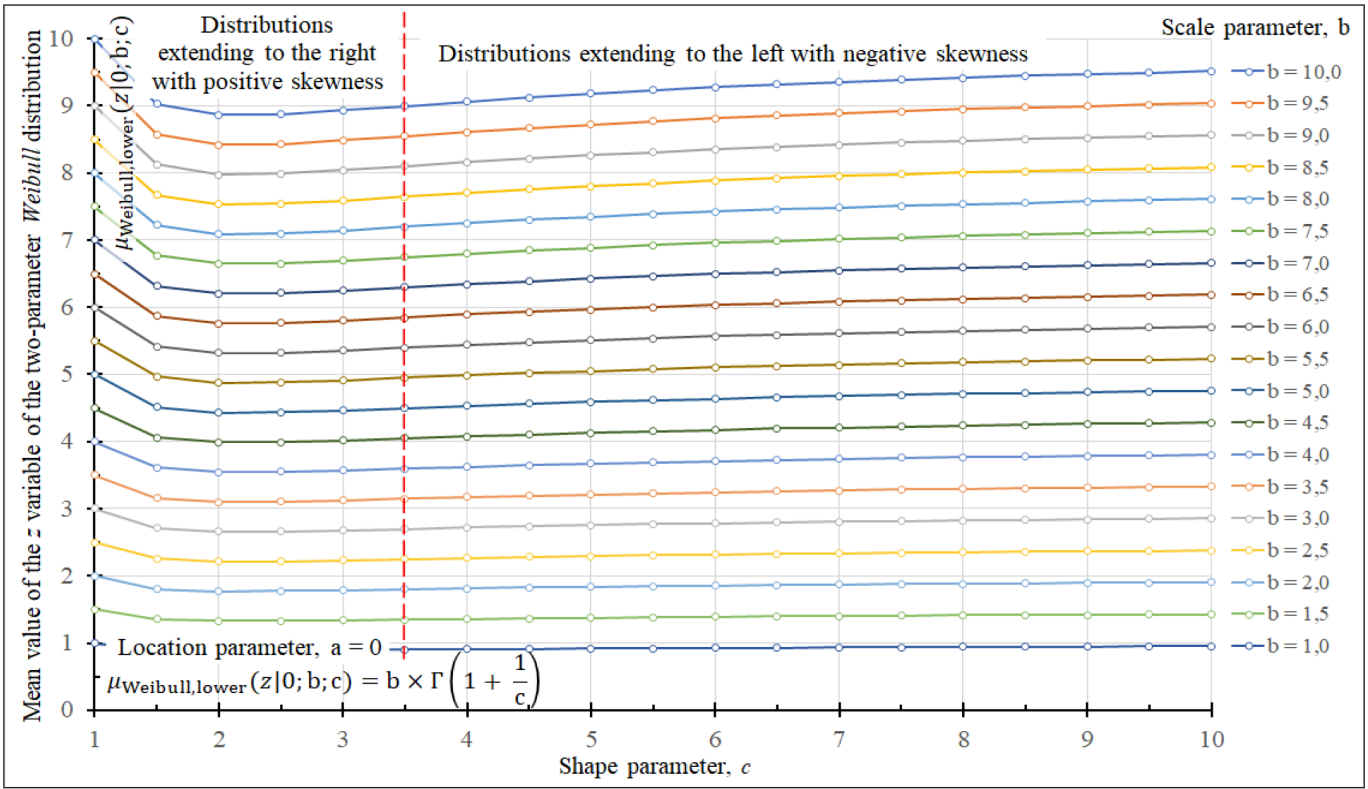


Figure 7: Expected mean value of the *z* independent variable of the two-parameter lower Weibull distribution semi-simplified by the location parameter „a" as a function of the scale parameter „b" and shape parameter „c", if 10.0 ≤ b ≤ 50.0 and if 1.0 ≤ c ≤ 10.0

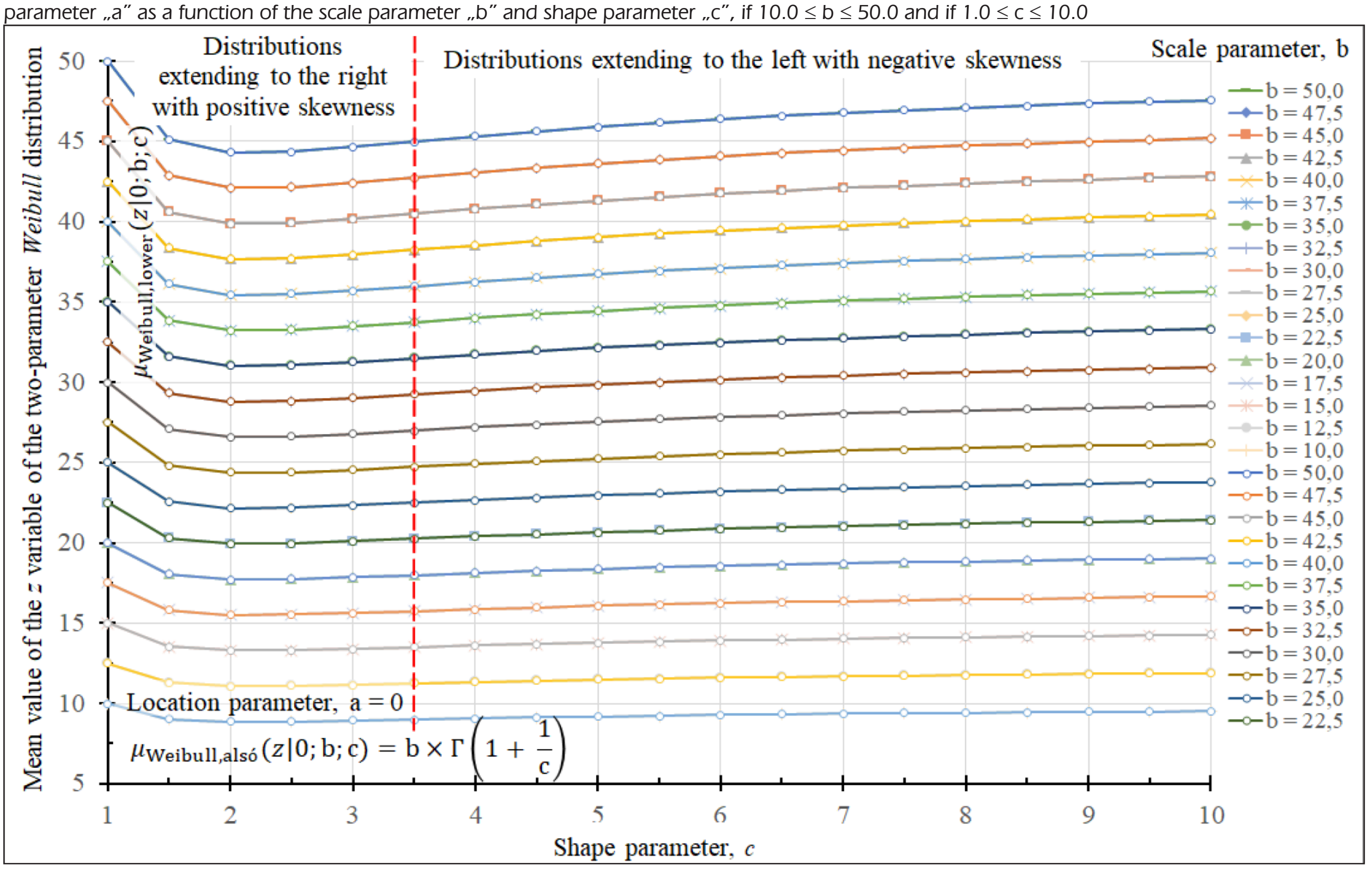


Figure 8: Nomogram for determination of parameters to the lower Weibull distribution

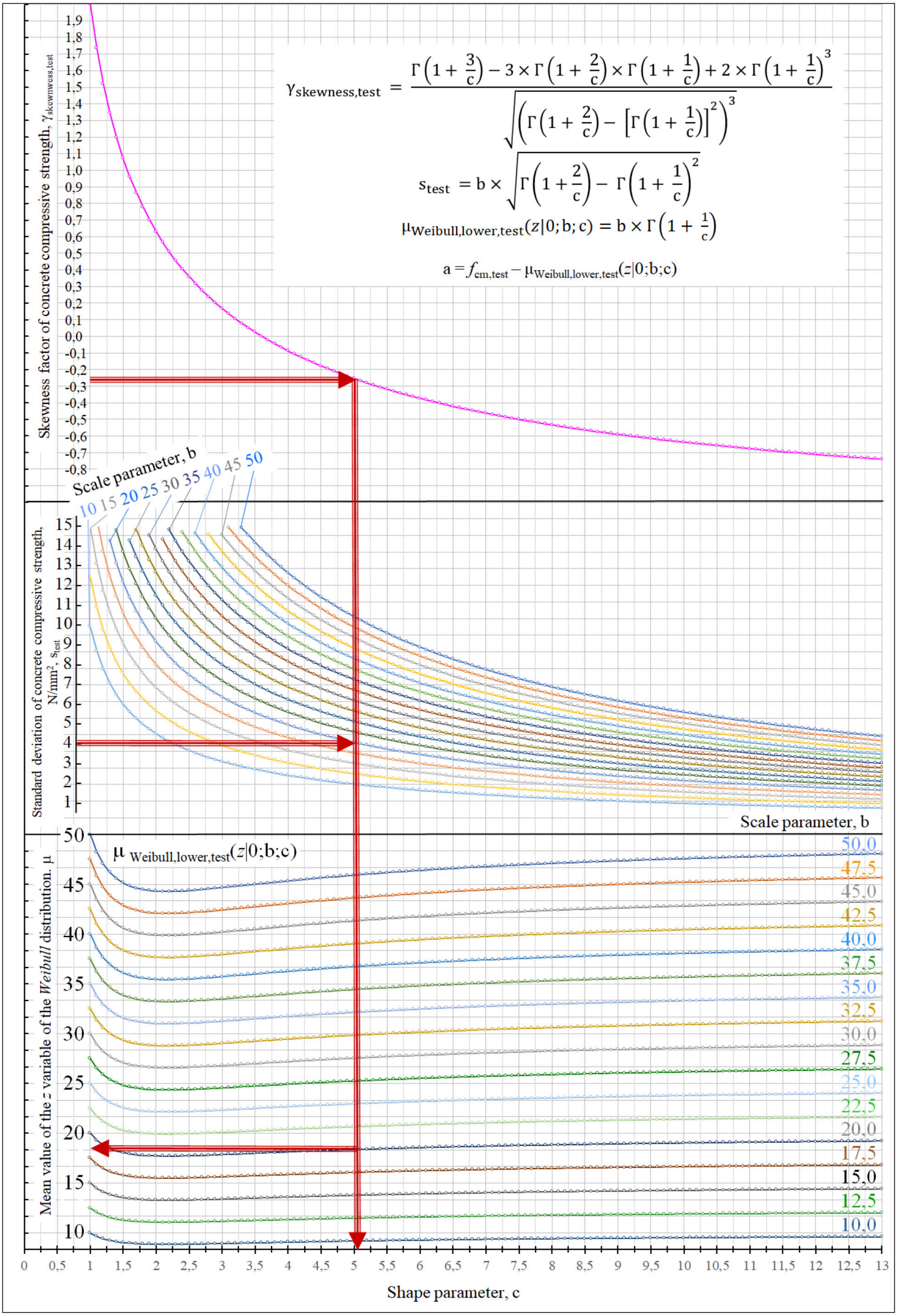


Figure 9: Demonstration for the solution process of the numerical examples in a nomogram

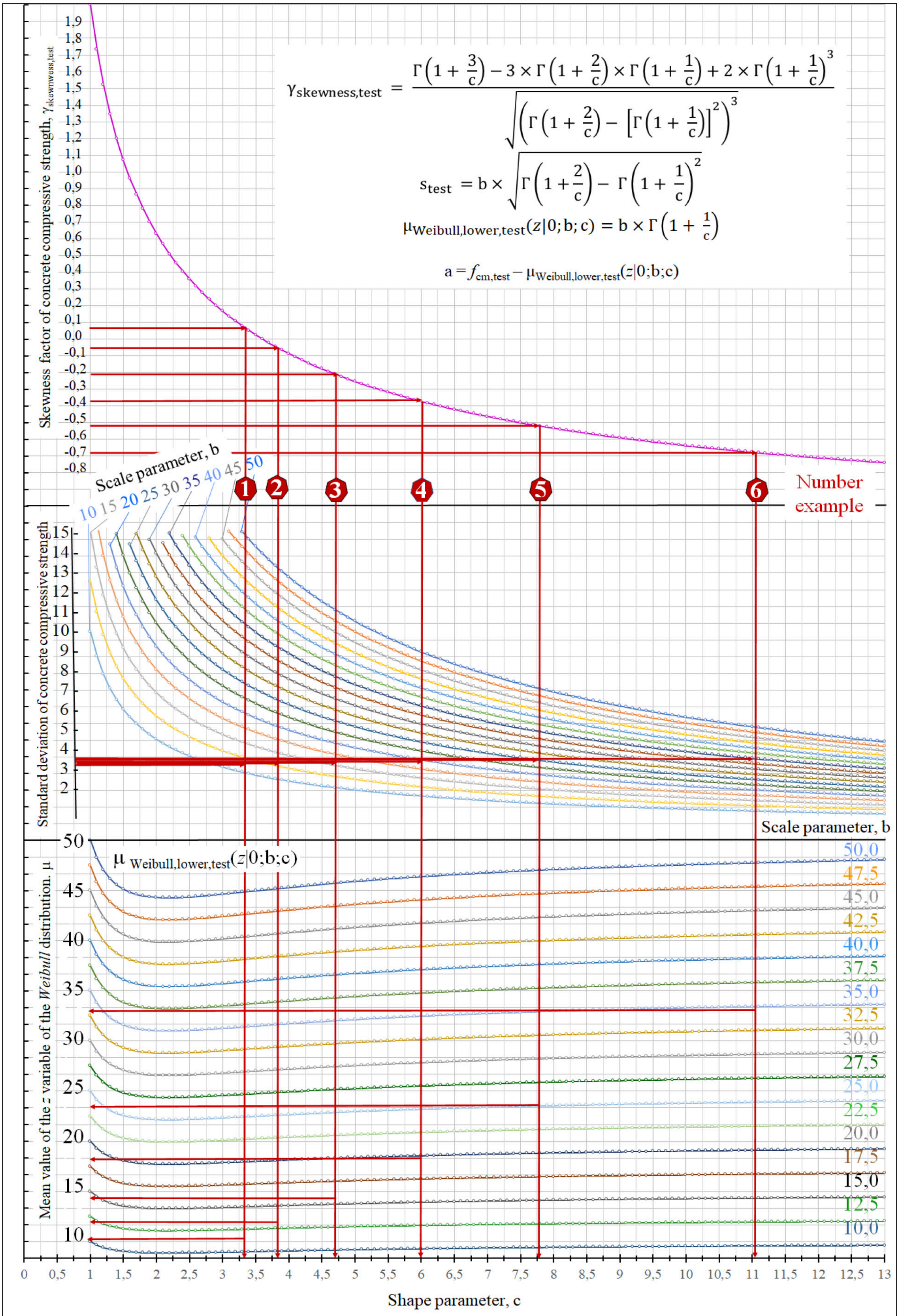


Table 2: Numerical examples for application of the lower Weibull extreme value distribution in case of concrete strength evaluation

SIGN OF THE NUMERICAL EXAMPLE					
1.	2.	3.	4.	5.	6.
INITIAL DATA					
Concrete compressive strength class, EN 1992-1-1:2004+A1:2014 Eurocode 2					
C16/20	C20/25	C25/30	C30/37	C35/45	C40/50
Empirical average of concrete compressive strength, N/mm², $f_{cm,cyl,test}$ EN 1992-1-1:2004+A1:2014 Eurocode 2					
24	28	33	38	43	48
Empirical standard deviation of concrete compressive strength, N/mm², s_{test}					
3,40	3,45	3,50	3,55	3,60	3,65
Empirical relative standard deviation of concrete compressive strength, $s_{rel,test}$					
0,142	0,123	0,106	0,093	0,084	0,076
Empirical skewness factor of concrete compressive strength, $\gamma_{skewness,test}$					
+0,07	-0,05	-0,21	-0,37	-0,52	-0,68
RESULT OF THE CALCULATION					
Mean value of the single-parameter simplified *Weibull* distribution,					
0,897	0,904	0,915	0,927	0,941	0,955
Mean value of the two-parameter semi-simplified *Weibull* distribution,					
10,271	11,802	14,436	18,230	23,660	33,430
Location parameter „a" of the *Weibull* distribution					
13,729	16,198	18,564	19,770	19,340	14,570
Scale parameter „b" of the *Weibull* distribution					
11,445	13,054	15,780	19,656	25,157	34,992
Shape parameter „c" of the *Weibull* distribution					
3,329	3,822	4,698	5,968	7,787	11,080
Empirical characteristic value of concrete compressive strength with *Weibull* distribution (5% quantile), N/mm², $f_{ck,test}$					
18,419	22,199	26,939	31,720	36,519	41,334
Empirical acceptance distance of concrete compressive strength with *Weibull* distribution, N/mm², $f_{cm,test} - f_{ck,test}$					
5,581	5,801	6,061	6,280	6,481	6,666
Acceptance constant of according to the number example (acceptance distance / standard deviation)					
1,641	1,681	1,732	1,769	1,800	1,826

compressive strength obtained from the measurements (s_{test}) are taken into account and the value of the scale parameter b of the *Weibull* distribution pertinent to the empirical standard deviation is found by implicit iteration:

$$s_{\mathrm{Weibull,low}}(z|0;\mathrm{b};\mathrm{c}) = \mathrm{b} \times \sqrt{\Gamma\left(1+\frac{2}{\mathrm{c}}\right) - \Gamma\left(1+\frac{1}{\mathrm{c}}\right)^2} = s_{\mathrm{test}}$$

Upon the successive iteration, calculation of skewness factor of the two-parameter lower *Weibull* distribution is repeated with input of different b values until a best value approaching the empirical skewness factor (e.g. to three decimals) is obtained. *Figures 2 and 3* can assist this procedure. These figures show diagrams of standard deviation for the *z* independent variable of the semi-simplified two-parameter lower *Weibull* distribution as a function of the scale parameter „b" and the shape parameter „c".
Figures 4 and 5 might also be of help: here diagrams for the *z* independent variable of the semi-simplified two-parameter lower *Weibull* distribution are shown: the standard deviation/ scale parameter quotient

$$\sqrt{\Psi_c} = \frac{s_{\mathrm{Weibull,low}}}{\mathrm{b}} = \sqrt{\Gamma\left(1+\frac{2}{\mathrm{c}}\right) - \left[\Gamma\left(1+\frac{1}{\mathrm{c}}\right)\right]^2}$$

and the square of the standard deviation/scale parameter quotient

$$\Psi_c = \left(\frac{s_{\mathrm{Weibull,low}}}{\mathrm{b}}\right)^2 = \left(\Gamma\left(1+\frac{2}{\mathrm{c}}\right) - \left[\Gamma\left(1+\frac{1}{\mathrm{c}}\right)\right]^2\right)$$

Calculated value of the shape parameter „c" can be refined by the use of the scale parameter „b", if required.

- Step 3: Mean value of $\mu_{\mathrm{Weibull,low}}(z|0;b;c)$ for the two-parameter semi-simpified lower *Weibull* distribution is calculated by implicit iteration using the (already known) values of the scale parameter „b" and shape parameter „c" :

$$\mu_{\mathrm{Weibull,low}}(z|0;\mathrm{b};\mathrm{c}) = \mathrm{b} \times \Gamma\left(1+\frac{1}{\mathrm{c}}\right)$$

Figures 6 and 7 show diagrams of the mean value $\mu_{\mathrm{Weibull,low}}(z|0;b;c)$ for the two-parameter semi-simplified lower *Weibull* distribution, depicted as a function of scale parameter „b" and shape parameter „c".

Figure 10: Diagrams of Step 1 of the solution for the numerical examples. Curves for the single-parameter generalized (GEV) density function for the lower Weibull distribution

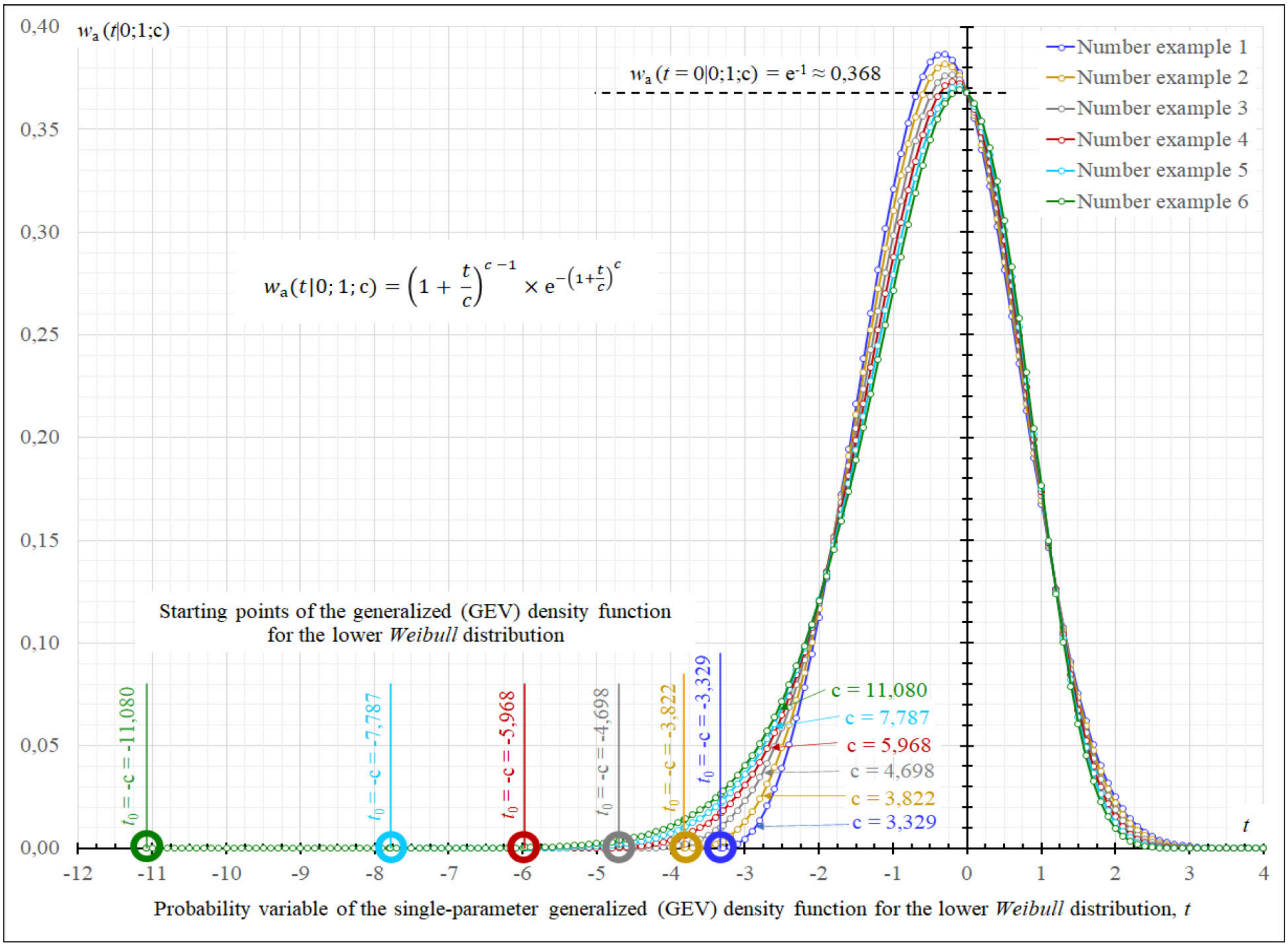


Figure 11: Diagrams of Step 1 of the solution procedure for the numerical examples. Curves of the single-parameter simplified density function for the lower Weibull distribution

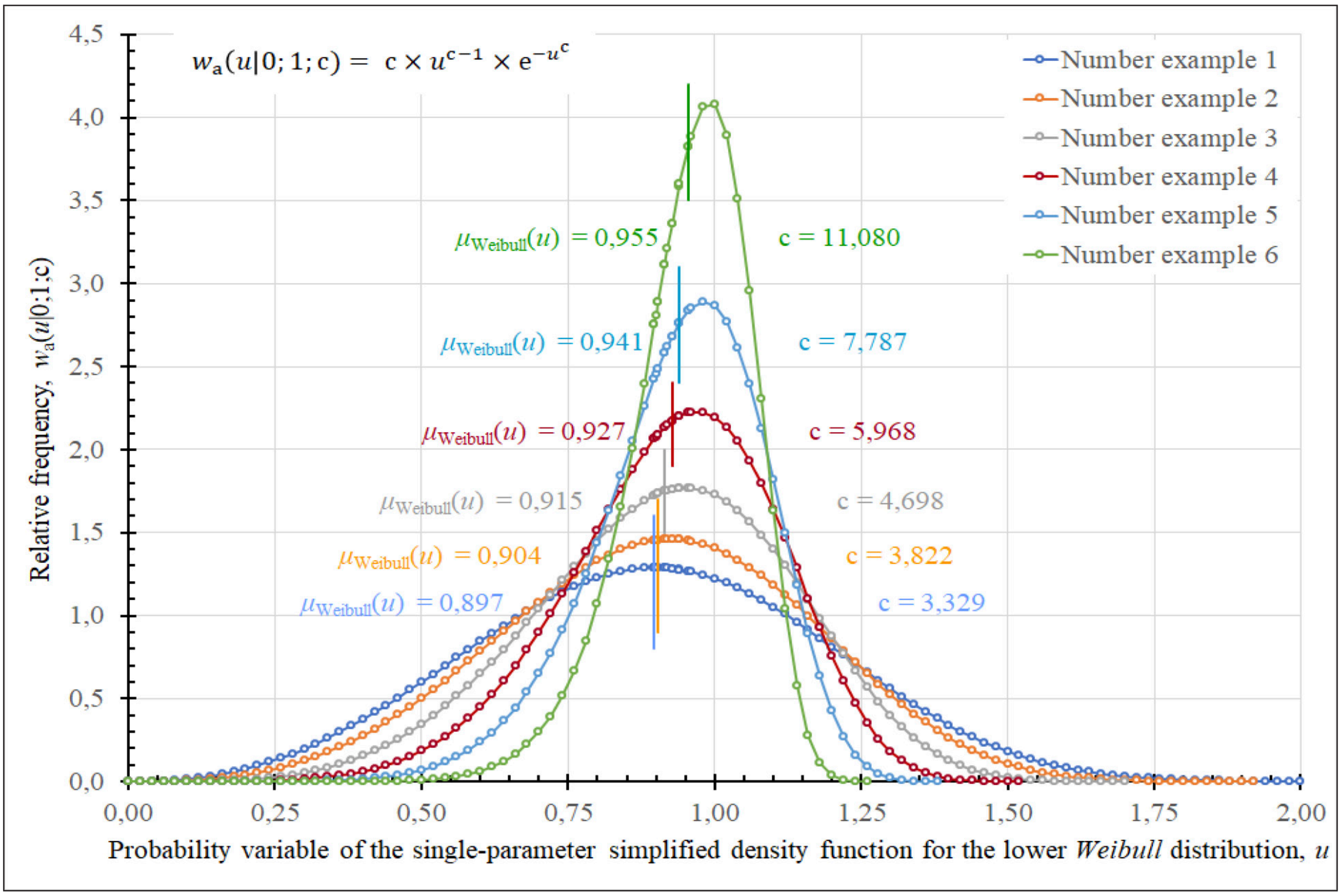
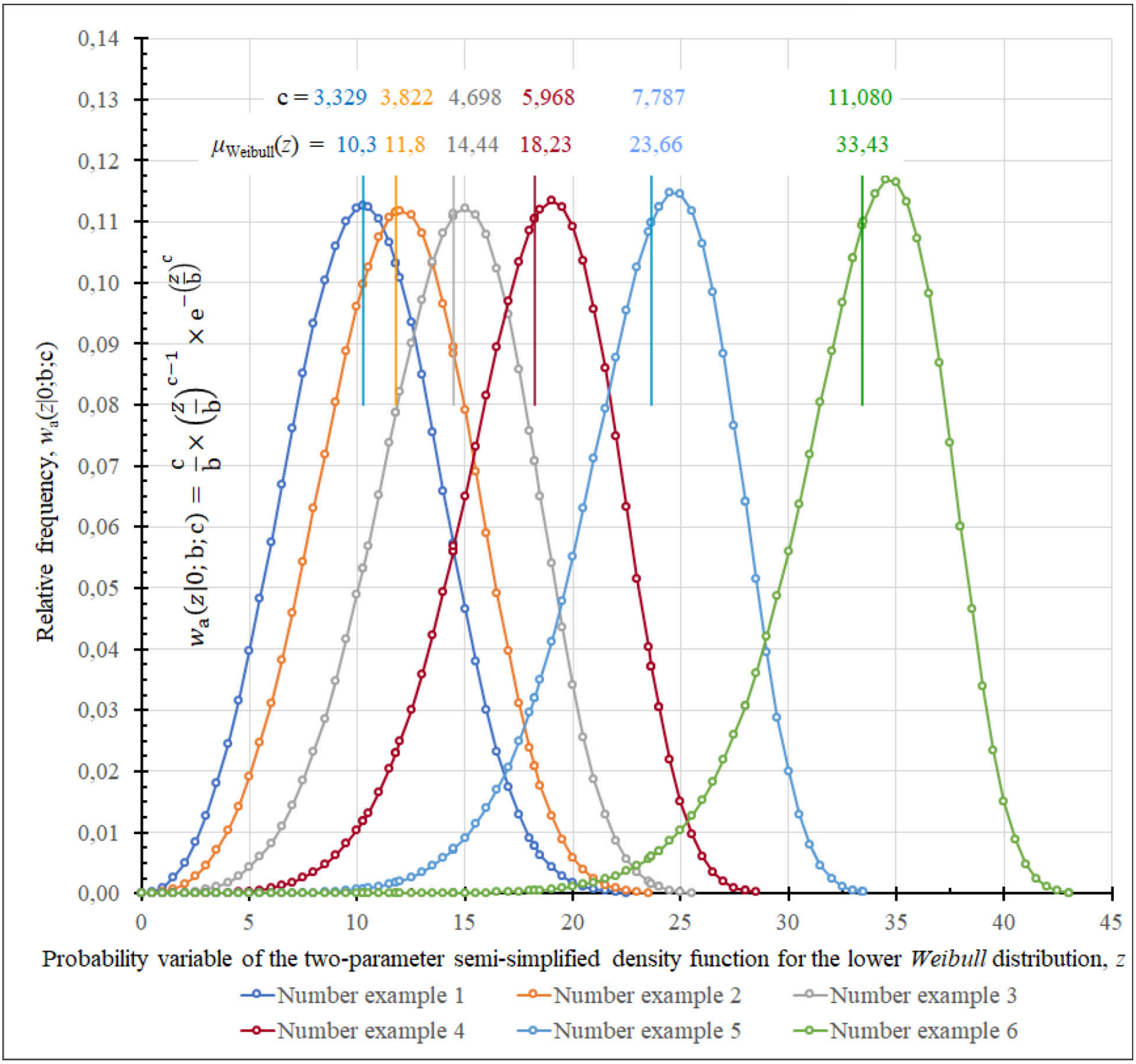


Figure 12: *Diagrams for Step 2 of the solution procedure for the numerical examples. Curves of the two-parameter semi-simplified density function for the lower Weibull distribution*



Value of the location parameter „a" can be obtained, when (corresponding to our basic assumption) the calculated mean value $\mu_{Weibull,low}(z|0;b;c)$ of the two-parameter semi-simplified lower *Weibull* distribution is subtracted from the empirical average value of the concrete compressive strength ($f_{cm,test}$):

$$a = f_{cm,test} - \mu_{Weibull,low}(z|0;b;c)$$

- Step 4: A nomogram (*Figure 8*) can be constructed using *Figures 1 - 7* in which calculated parameters of the *Weibull* distribution can be presented in their relations.

5. NUMERICAL EXAMPLE FOR EVALUATING THE COMPRESSIVE STRENGTH OF CONCRETE WITH WEIBULL DISTRIBUTION

Numerical examples have been solved by the calculation method of parameters to the *Weibull* distribution as recommended above. Initial data of the numerical examples are: empirical mean value of the concrete compressive strength, its empirical standard deviation and empirical skewness factor. Major results of the calculation are characteristics of the two-parameter *Weibull* distribution: the mean value, the values of the location parameter, the scale parameter, the shape parameter, as well as the empirical characteristic (5% fractile) value and the empirical acceptance distance of the concrete compressive strength following the *Weibull* distribution and its empirical acceptance distance. The initial data of the numerical examples based on older experimental results and result of the calculation are shown in Table 2. Solution process of the numerical examples is shown in *Figure 9*.

Curves have been drawn for the following cases applying the calculated data of the numerical examples: single-parameter generalized extreme value distribution (GEV) density function for the lower *Weibull* distribution (*Figure 10*), single-parameter simplified density function of the same (*Figure 11*), two-parameter semi-simplified density

Figure 13: Diagrams for Step 3 of the solution procedure for the numerical examples. Curves of the three-parameter simplified density function for the concrete compressive strength with the lower Weibull distribution

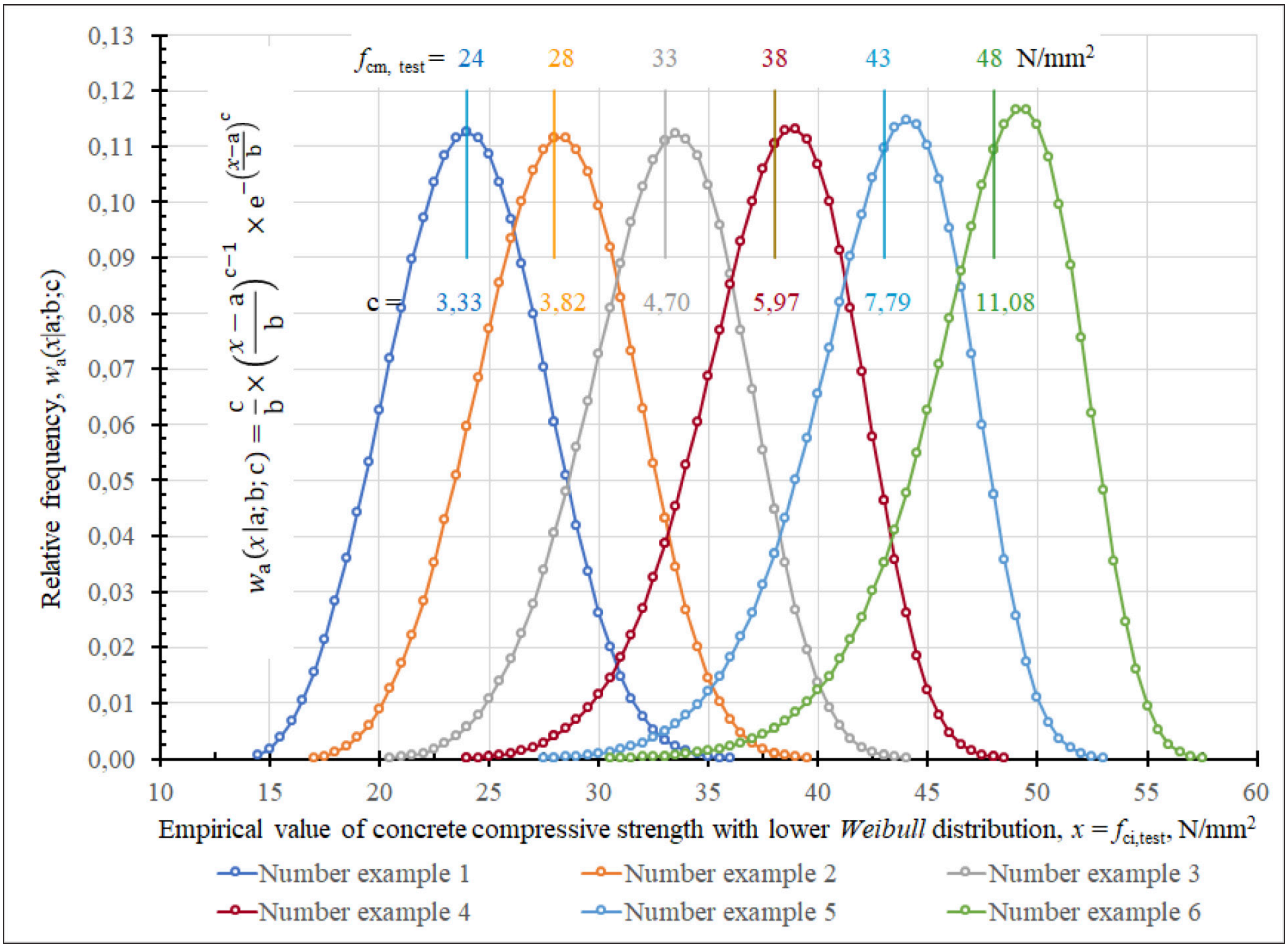
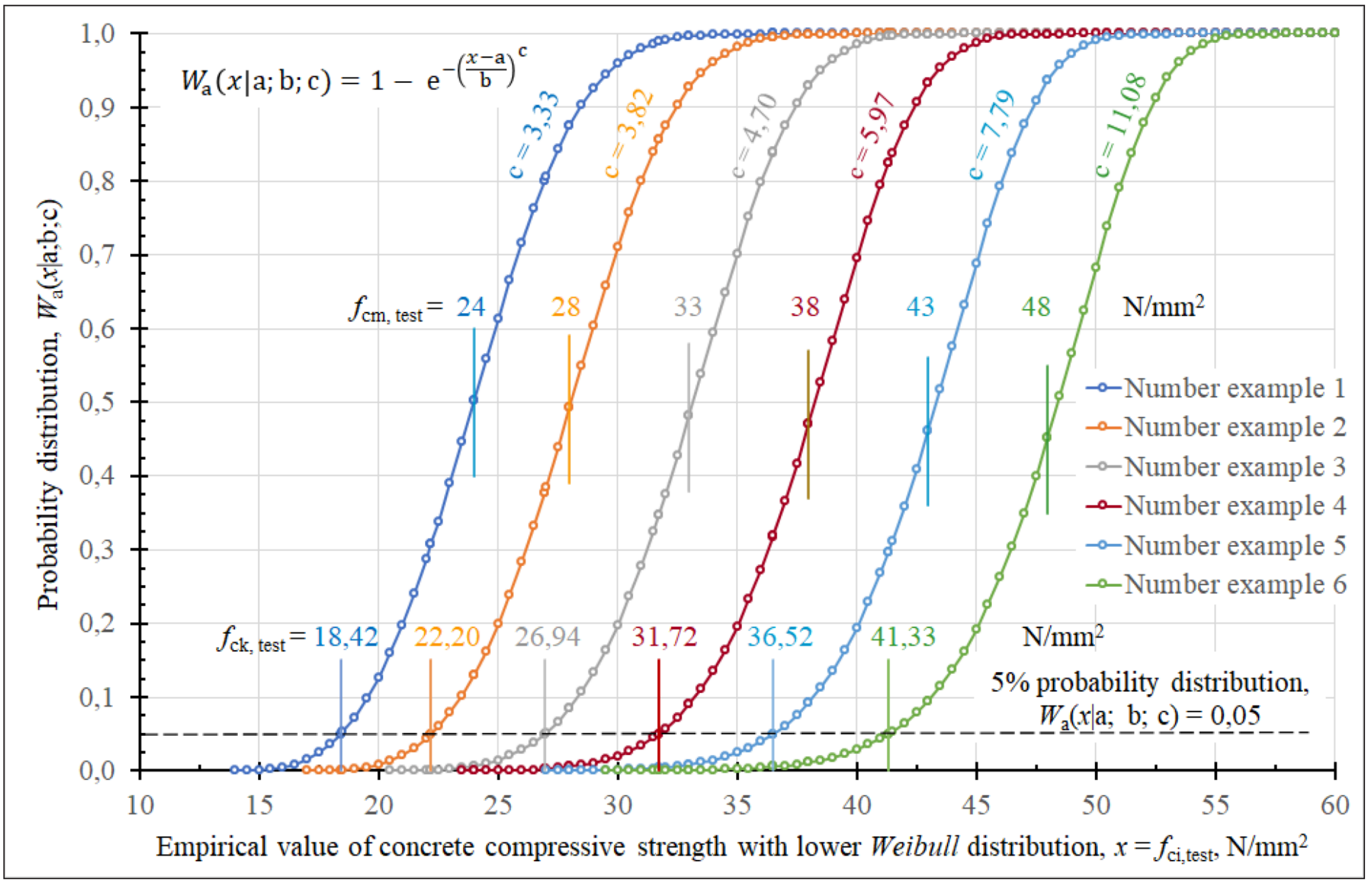


Figure 14: Diagrams for Step 3 of the solution for the numerical example. Curves of the three-parameter simplified distribution function for the concrete compressive strength with lower Weibull distribution



function of the same (*Figure 12*), as well as three-parameter density function and distribution function of the concrete compressive strength characterized with lower *Weibull* distribution (*Figures 13 and 14*, resp.).

The 5% empirical characteristic value for the concrete compressive strength represented with lower *Weibull* distribution was also shown in *Figure 14*. Empirical acceptance distance of the compressive strength ($f_{cm,test}$ – $f_{ck,test}$) is the difference of the empirical average ($f_{cm,test}$) and the empirical characteristic value ($f_{ck,test}$). Values of the difference in our numerical examples exceed 5.5 N/mm^2, in classes beyond C25/30 6 N/mm^2 (*Table 2*).

If the number n of the tested specimens is also considered, the acceptance distance is to be multiplied by the t_n/1,645 quotient, where t_n is *Student*'s t-value (*Federighi,* 1959), (*Stange* et al., 1966). Value of the quotient for a test size with n = 3 is e.g. 2.92/1.645 = 1.775, for a test size with n = 5 it is 2.132/1.645 = 1.296, for a test with n = 15 it is 1.761/1.645 = 1.071 or for a test with n = 35 it is 1.691/1.645 = 1.028.

6. CONCLUSIONS

1. Based on the last but one row of Table 2 we can conclude that the acceptance distance of the concrete compressive strength (performing a really skew distribution) calculated with the lower Weibull distribution is much higher than 4 N/mm^2 , as calculated corresponding to 8.2.1.3.2./Method A section of the EN 206:2013+A2:2021 standard pertinent to the initial production, and higher than 1.48×σ_{test} N/mm^2 corresponding to 8.2.1.3.2./Method B section (continuous production) of the same standard.
2. Consequetly, that according to the EN 206:2013+A2:2021 and the subsequent Hungarian product standard MSZ 4798:2016 with certain concrete compressive strength class (e.g. C40/50) has actually a lower average compressive strength – which is the basis for design of concrete composition – which takes into account the average compressive strength of the standards Eurocode (EN 1990:2002+A1:2005) and Eurocode 2 (EN 1992-1-1:2004+A1:2014, EN 1992-2:2005, EN 1992-3:2006) for the same concrete compressive strength class (Kausay et al., 2007).
3. This already unfavorable difference in acceptance factors (e.g. use of 1.48 instead of 1.645) for concretes with higher compressive strengths increases in reality, as we have shown in this publication using the Weibull distribution according to the actual slope (skewness) of the concrete strengths.
4. Evaluation of the concrete compressive strength results with the lower Weibull distribution in the concrete standard EN 206:2013+A2:2021 would serve the safety and durability of our concrete and reinforced concrete structures. Our antecedents who used the k, skewness factor depending on the average compressive strength in the k×t_n×s product (expressing the acceptance distance) as it was discribed in the withdrawn MSZ 4720-2:1980 Hungarian standard valid until August 2004. This factor k served an approximate consideration of skewness.

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8. REFERRED STANDARDS

EN 206:2013+A2:2021 „Concrete. Specification, performance, production and conformity"

EN 1990:2002+A1:2005 "Eurocode. Basis of structural design"

EN 1992-1-1:2004+A1:2014 „Eurocode 2: Design of concrete structures. Part 1-1: General rules and rules for buildings"

EN 1992-2:2005 „Eurocode 2: Design of concrete strucutres. Part 2: Concrete bridges. Design and detailing rules"

EN 1992-3:2006 „Eurocode 2: Design of concrete strucrures. Part 3: Liquid retaining and containment structures"

MSZ 4798:2016 „Concrete. Specification, performance, production, conformity, and rules of application of EN 206 in Hungary"

MSZ 4720-2:1980 „Quality control of concrete. Control of general characteristics" Withdrawn Hungarian standard

Hon. Prof. Tibor KAUSAY (1934), bachelor of civil engineering (1961), reinforced concrete engineer (1967), PhD (1969), candidate of Technical Sciences (1978), Ph.D. (1997), associate professor (1985), honorary professor at the Budapest University of Technology Department of Civil Engineering (2003), member of the Hungarian Section of the fib (2000), commemorative medal of Count Menyhért Lónyay of the Hungarian Academy of Sciences (2003), holder of the László Palotás Prize (Hungarian Section of the fib, 2015). Its activities cover research, development, expertise, education and standardization in the concrete technology and stone and gravel industries. The number of his publications is about 220.