STOCHASTIC DISTRIBUTION OF STRUCTURAL RESISTANCE OF REINFORCED CONCRETE BEAMS

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SUMMARY

In this paper, an analysis of the stochastic distribution of structural resistance of a reinforced concrete one-span girder is presented. The stochastic characteristics of the structural resistance are determined with three different methods: second-moment analysis, Monte-Carlo simulation and stochastic finite element method. The results derived from the different methods are compared with each other and also with the structural resistance calculated according to EUROCODE 2.

Keywords: reinforced concrete beams, safety, Monte-Carlo simulation, stochastic FEM, structural resistance

1. INTRODUCTION

Our research is related to safety of reinforced concrete beams subjected to combined stress. An appropriate method for solving large mechanical problems involving uncertainties is the stochastic finite element method (SFEM), hence we choose this method for the numerical safety analysis. We have implemented the SFEM on a beam model considering the material behaviour of the reinforced concrete. On the basis of the theoretical model, a computer program for calculating the safety of reinforced concrete beams subjected to compression, biaxial bending and torsion has been developed. The equations related to SFEM are formed in such way that globally acting forces can be compared with structural resistance at system level that makes it possible to judge the safety of a complex structure with one numerical value.

Beside the evaluation of safety of reinforced concrete beams, we also want to examine the applicability of the implied method. The purpose of this paper is to present an examination related to validity of probability distribution supposed by the applied SFEM. The knowledge of the distribution of structural resistance is essential when determining the safety of a structure. The SFEM generally provides the first two moments of the structural resistance only, and its distribution is supposed to be normal. We have made a comparison between the distributions obtained from SFEM analysis and Monte-Carlo simulation. In the paper, the results of this comparison are presented. The stochastic finite element analysis has been carried out on a reinforced concrete beam subjected to uniaxial bending, using 1D beam finite element model.

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2. ANALYSED STRUCTURE

2.1 Structural dimensions and material properties

The computations were carried out on a simple reinforced concrete one-span girder with a box section and uniformly distributed load (Fig. 1).

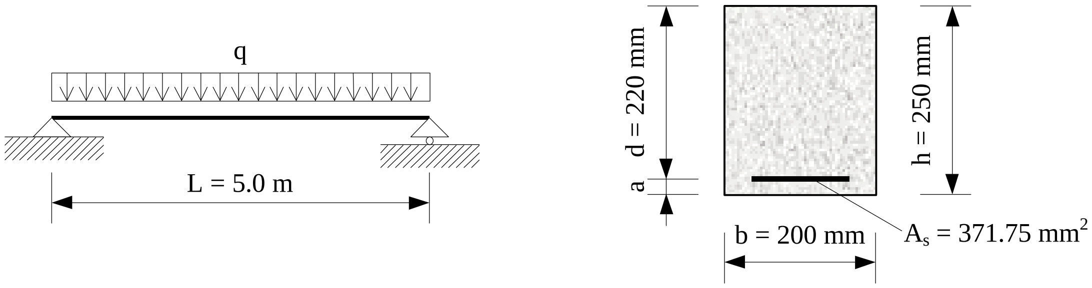

Fig. 1 Figure and cross section of analysed reinforced concrete one-span girder

Tab. 1 presents the most important material properties of the applied concrete and the reinforcement according to EUROCODE 2.

Concrete		**Reinforcing steel**	
C40/50		B500B	
f_{ck} [N/mm^2]	40	f_{yk} [N/mm^2]	500
$f_{cd} = f_{ck}/\gamma_c$ [N/mm^2]	26.67	$f_{sd} = f_{yk}/\gamma_s$ [N/mm^2]	434.8
ε_{cu} [‰]	3.5	ε_{su} [‰]	35
α	0.85	E_s [N/mm^2]	200000

Tab. 1 Material properties of concrete and reinforcing steel

In the calculations, a parabolic stress-strain relation for the concrete and a bilinear (elastic-plastic) stress-strain relation for the reinforcing steel was used (Fig. 2).

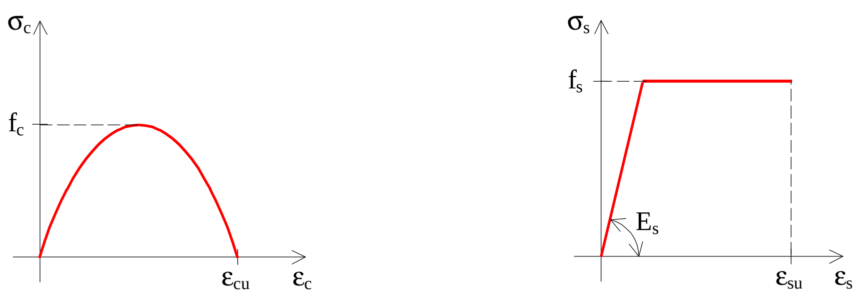

Fig. 2 Stress-strain diagrams for concrete and reinforcing bars

The stress-strain relation for the concrete presented in Fig. 2. was proposed by M. P. Collins (Szalai, 1990) and it can be expressed as:

$$\sigma_c(\varepsilon_c) = 4 \cdot f_c \cdot \left[\frac{\varepsilon_c}{\varepsilon_{cu}} - \left(\frac{\varepsilon_c}{\varepsilon_{cu}} \right)^2 \right]$$

2.2 Stochastic characteristics of input parameters

During the stochastic analysis of structural resistance, the width (b) and height (h) of the cross section, the concrete cover (a) and the strength of concrete (f_c) and reinforcing steel (f_s) were treated as random variables. The standard deviations of cross sectional width and height were determined on the basis of the concerning permitted limit values given by EUROCODE 2. The standard deviation of concrete cover was derived from experiments carried out by J. Almási (Almási, 1987). The mean values and standard deviations of strength of concrete and reinforcement were computed according to specifications of EUROCODE 2, and the skewness of these parameters was taken from the Hungarian Standard (MSZ 4720-80). Stochastic properties of the input parameters are summarised in Tab. 2.

Parameter	Mean value	Standard deviation [%]	Skewness
b	200 mm	2.13	0
h	250 mm	2.19	0
a	30 mm	18	0
$\mathbf{f_c}$	48 N/mm^2	10	0.48
$\mathbf{f_s}$	554.75 N/mm^2	6	0.28

Tab. 2 Stochastic properties of the input parameters

2.3 Resistance of the structure according to EUROCODE 2

In this paper, the resistance of the structure (q_R) is interpreted as the maximum uniformly distributed load it can resist. To serve as a basis of comparison, the structural resistance was first computed according to EUROCODE 2:

$$q_R = 8 \cdot \frac{M_R}{L^2} = 8 \cdot \frac{f_{sd} \cdot A_s \left(4 \cdot d \cdot b \cdot \alpha \cdot f_{cd} - 3 \cdot f_{sd} \cdot A_s\right)}{4 \cdot b \cdot \alpha \cdot f_{cd} \cdot L^2} = 10.00 \text{ kN/m} \qquad (2.1)$$

3. APPLIED COMPUTATIONAL METHODS

The stochastic characteristics of the resistance of the given structure were computed using three different methods: second-moment analysis, Monte-Carlo simulation and stochastic finite element method.

3.1 Second moment analysis

First a simple second-moment analysis was carried out. Assuming that the structure will fail at the place of the maximum bending moment, the mean value of the structural resistance can be calculated similar to (2.1), using the mean values of the input parameters instead of their design values given by EUROCODE 2:

$$q_{Rm} = 8 \cdot \frac{f_{sm} \cdot A_s \left(4 \cdot \left(h_m - a_m\right) \cdot b_m \cdot f_{cm} - 3 \cdot f_{sm} \cdot A_s\right)}{4 \cdot b_m \cdot f_{cm} \cdot L^2}$$

The standard deviation of the structural resistance (s_{qR}) can be calculated by expanding it about its mean value by Taylor series (Belytschko, Liu, Mani, 1986; Handa, Andersson, 1975). Assuming that q_R is function of a ξ random variable, s_{qR} can be approximately expressed with the first order partial derivative of q_R as

$$s_{q_R} = \frac{\partial q_R}{\partial \xi} \cdot s_\xi \tag{3.1}$$

where s_ξ is the standard deviation of ξ. The higher order partial derivatives were neglected from (3.1), but this approximation has about 0.1 per cent error only in our case. According to (3.1), the standard deviation of the structural resistance can be finally expressed in the following form (Szalai, 1990):

$$s_{q_R} = \sqrt{\left(\frac{\partial q_R}{\partial b} \cdot s_b\right)^2 + \left(\frac{\partial q_R}{\partial h} \cdot s_h\right)^2 + \left(\frac{\partial q_R}{\partial a} \cdot s_a\right)^2 + \left(\frac{\partial q_R}{\partial f_c} \cdot s_{f_c}\right)^2 + \left(\frac{\partial q_R}{\partial f_s} \cdot s_{f_s}\right)^2}$$

3.2 Monte-Carlo simulation

The distribution of the resistance of the sample structure was determined by Monte-Carlo method in two different ways.

3.2.1 Monte-Carlo simulation concerning to the resistance of midst cross-section

First it was assumed that the structure will fail at the place of the maximum bending moment, and the stochastic characteristics of q_R was calculated similar to (2.1), using randomly generated input parameters (b, h, a, f_c, f_s). The simulation was carried out in two slightly different ways. Once the distribution of all random input parameters were supposed to be normal, than according to experiences of E. Mistéth (Mistéth, 1974) the distributions of the structural dimensions (b, h, a) were assumed to be normal and the strengths of concrete and reinforcing steel were assumed to follow Gamma distribution. Normally distributed random numbers were generated the following way (Deák, 1986):

$$\xi_i^n = \xi_m + s_\xi \cdot \sqrt{-2 \cdot \ln \zeta_1} \cdot \sin 2\pi\zeta_2 \tag{3.2}$$

where ξ_m and s_ξ are the mean value and standard deviation of the universe, ζ_1 and ζ_2 are uniformly distributed random numbers. Random numbers having Gamma distribution were produced using the following relation (Deák, 1986):

$$\xi_i^g = \frac{-\log \prod_{j=1}^{r_{int}} \zeta_j + \Gamma(r_{frac})}{\lambda} + \xi_0$$

where r, λ and ξ_0 are the parameters of the Gamma distribution (r_{int} is the integer part of r and r_{frac} is the fraction part of r), ζ_j is a uniformly distributed random number and Γ is a randomly generated fraction-parameter Gamma function. The number of the simulations was one million in both cases.

3.2.2 Monte-Carlo simulation using finite element method

On the other hand, the Monte-Carlo method was also carried out using the finite element method. In this case, the distribution of the stochastic input parameters was supposed to be normal. Random input parameters were generated using (3.2) separately for each finite element. For the finite element analysis, 1D bar finite elements with deformations of order three were used (Bojtár, Gáspár, 1993). The beam was loaded with single-parameter load, witch means that the vector of external loads (**q**) can be expressed as a product of the load-intensity (q) and a load-distribution vector (**Φ**):

$$\mathbf{q} = q \cdot \mathbf{\Phi}$$

For each realisation of the input parameters the structure was loaded until failure and the maximum value of the load-intensity was interpreted as the structural resistance (q_R). The number of the simulations was in this case two hundred only because the process was too consumptive of time.

3.3 Stochastic finite element method

The stochastic characteristics of the resistance of our structure were finally computed by the stochastic finite element method. The mean value of the structural resistance was determined by finite element method as described in 3.2.2. using the mean values of the input parameters (b_m, h_m, a_m, f_{cm}, f_{sm}). The standard deviation of the structural resistance was calculated using the stochastic finite element method from the following equation (Handa, Andersson, 1975; Eibl, Schmidt-Hurtienne, 1995; Koris, 1996):

$$\mathbf{C}_q = \delta\mathbf{q}_M \cdot \delta\mathbf{q}_M^T = \mathbf{K}_M^{-1} \cdot \frac{\partial \mathbf{K}}{\partial \xi} \cdot \mathbf{u} \cdot \delta\xi \cdot \mathbf{C}_\rho \cdot \delta\xi^T \cdot \mathbf{u}^T \cdot \frac{\partial \mathbf{K}^T}{\partial \xi} \cdot \mathbf{K}_M^{-T}$$

where $\mathbf{C}_q$ is the covariance matrix witch includes s_{qR}, **K** is the global stiffness matrix of the structure, **u** is the vector of nodal displacements, $\delta\xi$ includes the standard deviations of random input variables, $\mathbf{C}_\rho$ is the correlation matrix and $\mathbf{K}_M$ is:

$$\mathbf{K}_M = \begin{bmatrix} k_{1,1} & \cdots & k_{1,i-1} & -\Phi_1 & k_{1,i+1} & \cdots & k_{1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ k_{j-1,1} & \cdots & k_{j-1,i-1} & -\Phi_{j-1} & k_{j-1,i+1} & \cdots & k_{j-1,n} \\ k_{j,1} & \cdots & k_{j,i-1} & -\Phi_j & k_{j,i+1} & \cdots & k_{j,n} \\ k_{j+1,1} & \cdots & k_{j+1,i-1} & -\Phi_{j+1} & k_{j+1,i+1} & \cdots & k_{j+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ k_{n,1} & \cdots & k_{n,i-1} & -\Phi_n & k_{n,i+1} & \cdots & k_{n,n} \end{bmatrix} \qquad (3.3)$$

In (3.3) i is for the number of the node where the structure fails.

4. RESULTS

The stochastic characteristics of the structural resistance were calculated in ten different ways based on the methods described in chapter 3. First, the second-moment analysis was carried out and a normal distribution was fitted to the derived statistics (method 1). The results derived from Monte-Carlo simulation described in 3.2.1. with normally distributed random input parameters were directly analysed (method 2), than the stochastic characteristics of the structural resistance were also computed on the fitted normal distribution (method 3). The empirical probability density function of the structural resistance and the fitted theoretical normal distribution are shown in Fig. 3.

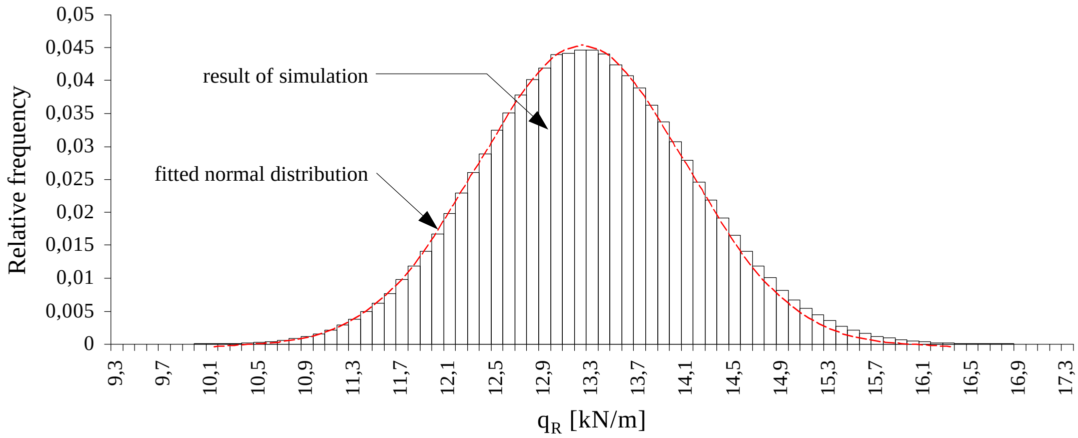

Fig. 3 Probability density function of the structural resistance derived from Monte-Carlo simulation with normally distributed random input parameters

Results of the simulation using Gamma-distributed random concrete and steel strengths as input parameters (3.2.1) were also directly analysed (method 4). To these results a normal distribution (method 5) and a Gamma distribution (method 6) was also fitted, since the empirical probability density function has showed positive skewness. The empirical and fitted theoretical probability density functions of the structural resistance derived from these methods are shown in Fig. 4.

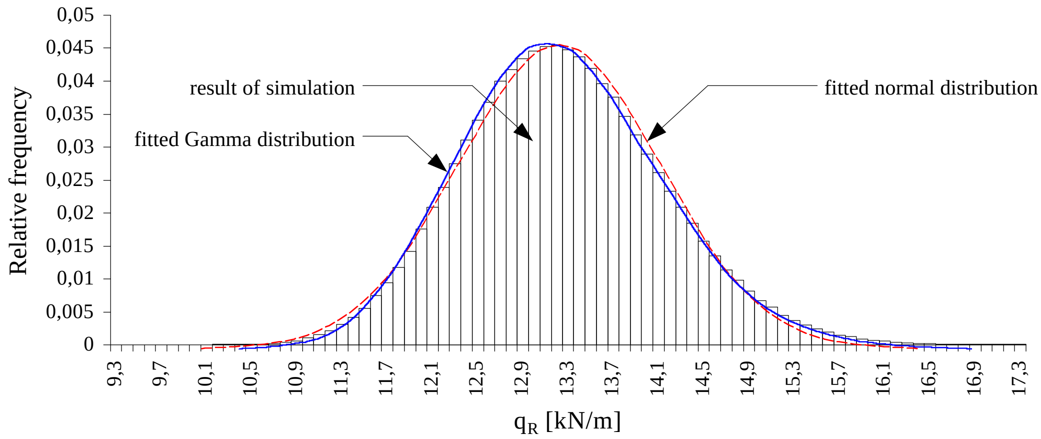

Fig. 4 Probability density function of the structural resistance derived from Monte-Carlo simulation with Gamma distributed random concrete and steel strengths

The mean value and standard deviation of the structural resistance (q_R) were also computed by stochastic finite element method, and a normal distribution was fitted to these statistics (method 7). Fig. 5 represents the relation between the load intensity (q) and the vertical deflection of midspan (y) calculated by stochastic finite element method. The stochastic characteristics of the load intensity calculated at the failure of the structure ($y = y_{max}$) are interpreted as the characteristics of the structural resistance.

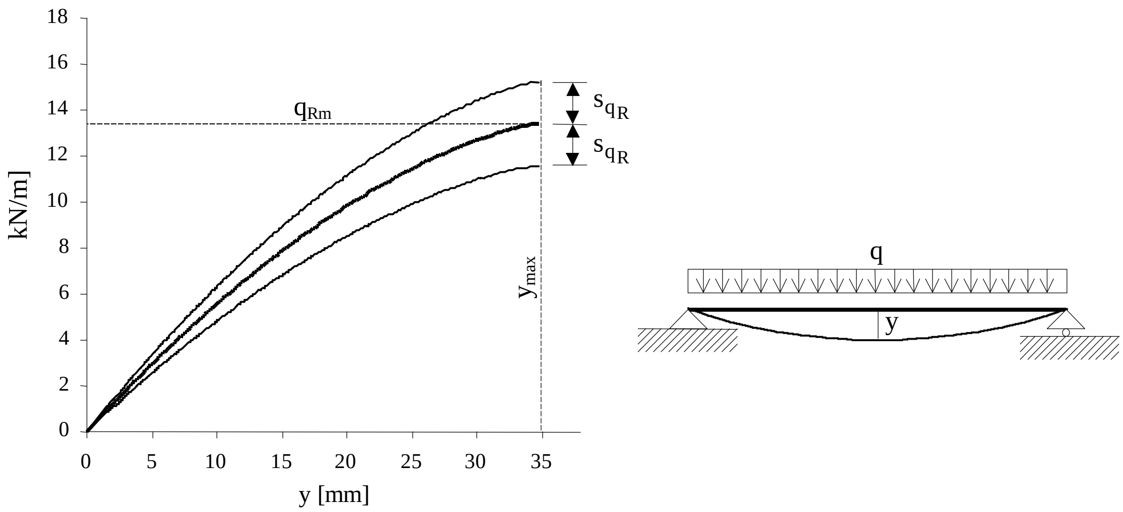

Fig. 5 Scatter of load-bearing capacity computed by stochastic finite element method

Finally a Monte-Carlo simulation using finite element method (3.2.2.) was carried out (method 8). To the results of the simulation a normal distribution (method 9) and a Weibull distribution (method 10) was also fitted, since the empirical probability density function seemed to have negative skewness. The resulting empirical and fitted theoretical probability density functions of the structural resistance are shown in Fig. 6.

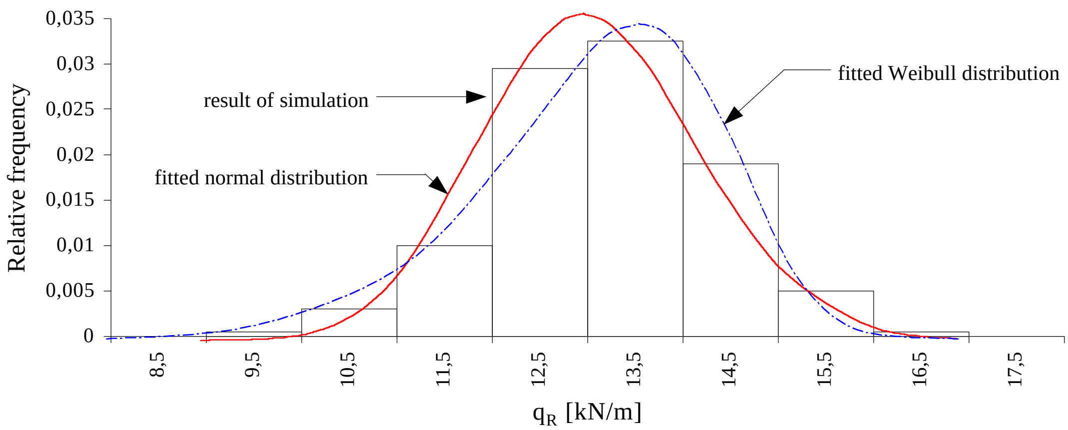

Fig. 6   Probability density function of the structural resistance derived from Monte-Carlo simulation using finite element method

Using the above methods the following statistics of the structural resistance were computed: mean value, standard deviation, skewness, 1‰ lower quantile, probability of the resistance being smaller than the resistance computed by EUROCODE 2 and finally the reliability of the estimation in case of theoretical distributions using the theorem of Kolmogorov and Smirnov (the closer 1-K(z) is to 1 the better the estimation is). The computed statistics of the structural resistance are listed in Tab. 3.

Method	**Mean value [kN/m]**	**Standard deviation [%]**	**Skewness**	**1 ‰ Quantile [kN/m]**	**p(q_R < 10 kN/m) [‰]**	**1 - K(z)**	**CPU time [hours]**
1	13.2741	6.71	0	10.5197	0.1198	-	-
2	13.2556	6.72	0.0852	10.6011	0.0750	-	0.48
3	13.2556	6.72	0	10.5016	0.1296	0.1307	-
4	13.2567	6.71	0.2703	10.8163	0.0060	-	3.12
5	13.2567	6.71	0.2703	10.5076	0.1257	0.0000	-
6	13.2567	6.71	0.2703	10.8449	0.0036	0.9979	-
7	13.4250	13.61	0	9.2693	4.0233	-	0.53
8	13.1540	8.77	-0.1605	-	-	-	13.63
9	13.1540	8.77	0	9.5854	3.1551	0.9733	-
10	13.1540	8.77	-0.1605	7.8438	20.2006	0.9728	-

Tab. 3 Statistics of the structural resistance (q_R) derived from different methods

5. CONCLUSIONS

Results obtained from stochastic finite element method were compared with the results of other procedures and it can be approved that this method it is accurate and efficient enough in safety analysis of reinforced concrete beams. Differences in the results concerning the statistics of the structural resistance are rather effected by the differences in evaluation methods of structural resistance . This process is less time-consuming than the Monte-Carlo simulation (see utilised CPU times in Tab. 3) and opposed to second-moment analysis and Monte-Carlo simulation (methods 1, 2 and 4) it can be easily used in case of more complex structures too.

6. REFERENCES

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