THE EFFECT OF FLEXURAL RIGIDITY ON THE MOMENT AND DEFLECTION OF STATICALLY INDETERMINATE REINFORCED CONCRETE ELEMENTS

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SUMMARY

Current procedures for computation of moment and deflection of statically indeterminate reinforced concrete beams are presented, and the results obtained are compared. Also, the procedures described are analysed concerning the complexity of their use. The purpose of the research is to work out a method, which can be used in engineering practice, for the calculation of moments and deflections of reinforced and prestressed concrete flat slabs.

Keywords: reinforced concrete beam, flexural rigidity, cracked state, rotation, deflection

1. INTRODUCTION

The importance of accuracy in the computation of the deformations of structures has grown recently, as the loadbearing structures of buildings tend to have greater spans, as smaller deformations and crack-widths are accepted due to aesthetic requirements, and also, as high-strength materials are widely used, resulting in greater strains.

Building codes propose different methods for the computation of deflection of reinforced concrete flat slabs. In a computer analysis, the classical methods of elasticity can be applied. A disadvantage of this method is that it assumes the moment of inertia of the slab constant, i.e. steel amount and cracks of the slab, which influence the inertia, are not accounted for.

In the commonly applied equivalent frame method, to simplify the calculation, the codes propose that deflection of continuous beams (representing column and middle strips) should be calculated. In some cases either the deflection of the column strip or that of the middle strip relative to the column strip has to be determined, and the total deflection of the centre point of a panel is the sum of these. The calculation starts with the determination of the moment along these beams, which can be obtained by the distribution of the total moment to column and middle strips according to a certain ratio given in the code (see Fig. 1). (The ratio applied here is given by ACI 318-89/13.6.4.)

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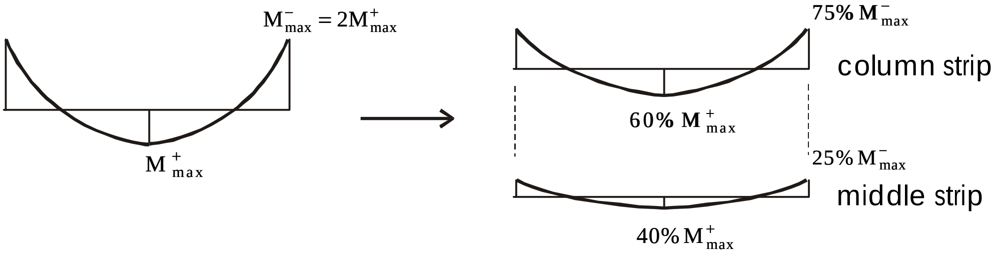
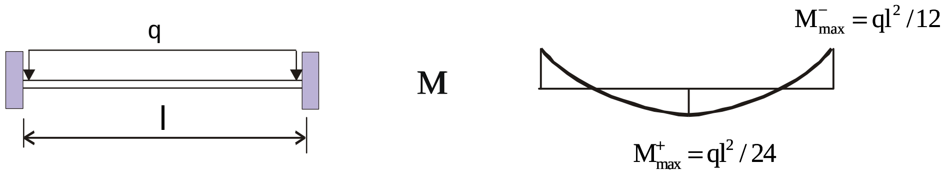
There are no instructions in the codes how to get the exact shape of the moment diagram (whether it is a parabola; location of zero points). Since the ratio of the maximum negative and positive moments in a span does not remain 2, assuming constant inertia along the beam would result in rotation at the supports, which does not occur in case of symmetrical loading, and this way the requirements of compatibility are not satisfied. This arises the question: What are the correct values of the moment of inertia to be used in deflection computation?

Fig. 1 Total moment; moment of column and middle strips



The static model of the column and middle strips of an internal span of a flat slab having several spans in both directions (so that the loading of an internal span can be assumed



symmetrical), all the panels loaded by uniformly distributed load, can be a beam fixed at both ends, loaded by uniformly distributed load q (see Fig.2). Since the amount of steel varies along the beam and cracks occur around the positive and negative maximum moments, the cross-sectional properties are not constant, and thus, the moment of inertia varies along the beam. This fact leads to a further problem: the maximum positive and negative moments calculated assuming constant moment of inertia ($ql^2/24$; $ql^2/12$) are not valid anymore.

Fig. 2 Moment diagram of a beam fixed at both ends

This paper discusses the problems: what procedures to apply to obtain the moment diagram and to determine the flexural rigidity in function of location for deflection computation.

2. ACCOUNTING FOR THE TENSION STIFFENING EFFECT IN STATE II

At locations where $|M|<M_{crk}$ the section remains uncracked, it is in State I, and its moment of inertia is I_I, the moment of inertia of the transformed section. The cracked segments of the beam are in State II, but I_{II} would underestimate the actual moment of inertia, as this way the tension stiffening effect of the concrete (the contribution of the concrete to the steel stress between the cracks) is ignored. To account for this effect, the ACI and the Eurocode-2 propose different methods. While the ACI proposes that an interpolation should be done between I_g and I_{crk} to get the effective moment of inertia, I_e, the Eurocode-2 suggests that a similar interpolation should be done between the deformation (curvature, rotation or deflection) in States I and II. According to A. Ghali,

the method proposed by the ACI is not accurate (A. Ghali, 1993), therefore, the other method was chosen. The deflection of a cracked member

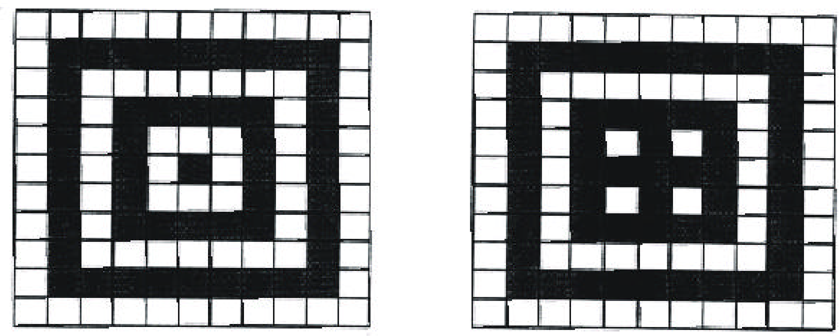
$$a = (1 - \zeta)\, a_I + \zeta\, a_{II} \qquad \text{(N.1)}$$

where

$$\zeta = 1 - ß\, (M_{crk} / M)^2 \qquad \text{(N.2)}$$

3. LOADING SCHEMES TO OBTAIN MAXIMUM DEFLECTION

In case of a slab having several panels in both directions and supported by beams, to obtain the maximum moment and deflection at the centre point of the central panel, a chessboard pattern loading scheme has to be applied: live load acts on alternate panels. This loading scheme does not provide maximum deflections in case of a flat slab, since, in terms of deformation, it acts like a catenary structure: positive load in a panel causes positive deflection in the adjacent panels. Finite element computer modelling (assuming constant moment of inertia throughout the slab) provides the following loading schemes to obtain maximum moment and deflection (see Fig.3):



M_{max} a_{max}

Fig.3 Loading schemes to obtain maximum moment and maximum deflection at the centre point of the central panel (supports at the intersections; no beams)

Considering that the column strips of a reinforced concrete flat slab have greater moment of inertia than the middle strips because of a greater steel ratio, so they act as beams, the behaviour of a flat slab must be somewhere in between that of a flat slab having constant moment of inertia and a slab supported by beams. An aim of the research is to find the loading scheme of a flat slab that provides the greatest deflection at a certain location of the slab.

4. PROCEDURES FOR THE COMPUTATION OF DEFLECTION OF REINFORCED CONCRETE BEAMS FIXED AT BOTH ENDS

4.1. Constant moment of inertia along the beam

As an approximate method, the moment of inertia of a beam (fixed at both ends, loaded by a uniformly distributed load q) can be assumed constant along the beam, and the deflection can be computed in State II (see Fig.4). Since the inertia is assumed to be constant, the maximum positive and negative moments: $M^+_{max} = ql^2/24$ and $M^-_{max} = ql^2/12$. To account for the tension stiffening effect, the final deflection can be obtained by an interpolation between a_{II} and a_I, where the latter is computed assuming I_I^+ constant. (I_I

is only slightly greater than I_g, thus, I_g may be used, ignoring the steel amount.) The application of this method is simple, it can be recommended even for manual computation.

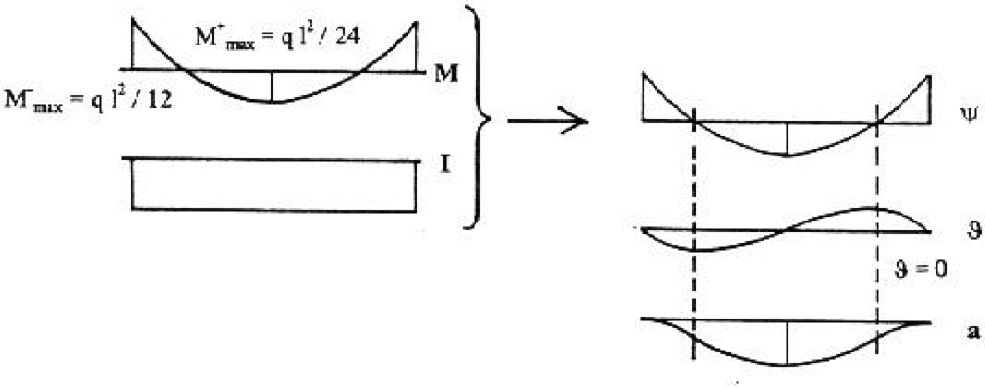


Fig.4 Moment, moment of inertia, curvature, rotation and deflection diagrams

Assuming that deflection computation is done by the virtual work method, the following moment diagrams have to be integrated graphically (see Fig.5). It can be seen in these diagrams that the midspan area of the first diagram has bigger influence on the deflection, since it will be multiplied by greater values of the second diagram. Therefore, the moment of inertia to be applied is the one that can be obtained from the steel amount at the maximum positive moment (I_{II}^+), at midspan.

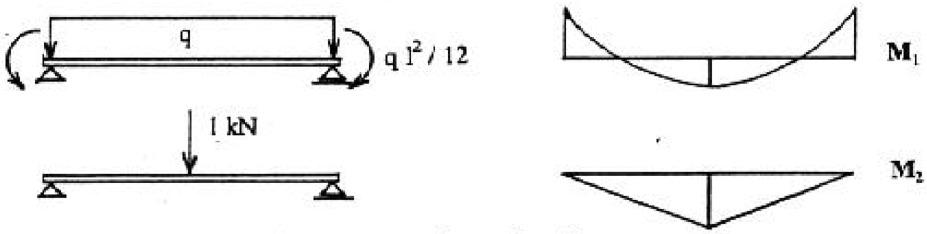


Fig.5 Moment diagrams to be integrated graphically

4.2. Approximation of real values of moment of inertia

Assuming that at the uncracked segments of the beam, the inertia is I_I, and that at the cracked segments, the amount of steel is proportional to the moment and that I_{II} is proportional to the amount of steel gives the following diagram for the moment of inertia (see Fig.6).

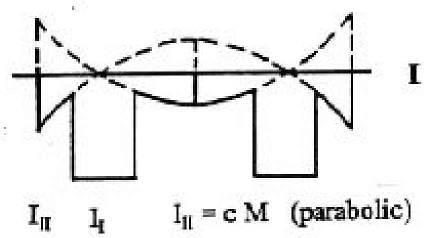


Fig.6 Moment of inertia dic

Computation of deflection using such a diagram for the inertia is possible only by computer: double integration of the M / E_cI function provides the deflection in function of location. The application of this method is complicated and there are reasons why it does not provide correct deflection values:

1/ the moment of inertia is not constant, and the original moment diagram (see Fig.1) is not valid any more,

2/ the location and length of the uncracked segment determined from the original moment diagram is not accurate, and

3/ the positive and negative areas of the curvature diagram ($M / E_c I$) are not equal, thus, the calculation will not provide zero rotation at the fixed ends.

4.3. Curvatures at three sections

A. Ghali proposed that in case of continuous beams, curvatures should be calculated at three sections: above the supports (in our case, at the fixed ends) (ψ_1; ψ_3) and at midspan (ψ_2), using the moment of inertia computed from the steel amount at the certain section, and the curvature diagram is assumed to be a parabola. This method does not require the application of a computer, since the following formula provides the deflection, the result of the double integration of the curvature diagram:

$$a = l^2 / 96\ (\psi_1 + 10\psi_2 + \psi_3) \qquad \text{(N.3)}$$

where $\psi_1 = \psi_3$ in our case because of symmetry. (A. Ghali, 1993) However, performing this method shows that the result overestimates the deflection, since there is positive rotation at the clamps according the calculation (see Fig.7).

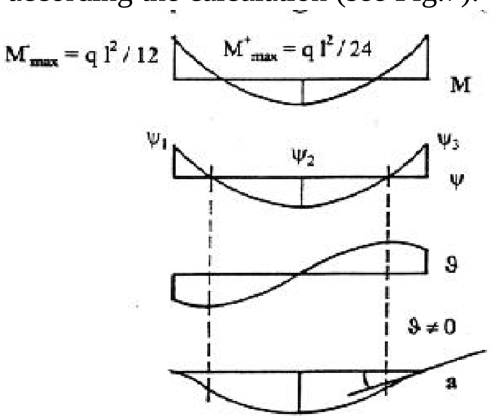


Fig.7 Moment, curvature, rotation and deflection diagrams

4.4. Constant curvature along the beam

Assuming that the steel amount is proportional to the moment, and that the moment arm of the resultants of stresses is constant along the beam, the stress in the steel is constant. At cracked sections, strain is a function of the steel stress, which results in constant strain along the beam, and thus, constant curvature, too. (The curvature can be calculated from the maximum positive moment and the moment of inertia at midspan.) The deflected shape of the beam consists of circle arcs. For geometric reasons, in the deflection diagram, the inflection points will be at the quarters of the span. For that, the moment diagram has to have zero points at these locations (see Fig.8). This geometric requirement and the total moment being $ql^2/8$ provides the following maximum moments:

$$M^-_{max} = 3\,q\,l^2 / 32 \quad \text{and} \quad M^+_{max} = q\,l^2 / 32 \qquad \text{(N.4)}$$

Deflection computation by this method can be performed manually; the following formula provides the midspan deflection:

$$a = \psi\, l^2 / 16 \qquad \text{(N.5)}$$

The effect of tension stiffening can be accounted for by computing a_I (assuming uncracked section; I_I) and a_{II} values, and interpolating between them by the earlier mentioned interpolation coefficient, ζ.

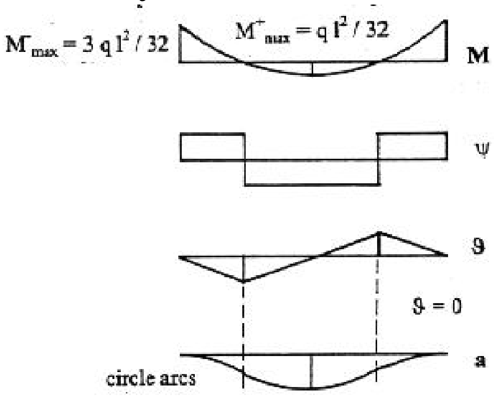


Fig.8 Moment, curvature, rotation and deflection diagrams

4.5. Finite element method

The service moment along the beam is function of the load and the stiffness (moment of inertia), but the moment has to be known for the computation of the moment of inertia at a certain section. (Knowing the moment at a section, the necessary amount of steel and then the moment of inertia of the transformed section can be computed.) By the finite element method, by some cycles of iteration (calculating moments and inertias of segments of the beam alternately) the final moments can be obtained. The method also provides deflection values, which can be regarded as the real deflection of the beam, since they are calculated using corresponding moment and moment of inertia diagrams. A disadvantage of this method is that the computation can only be performed by computer.

5. COMPARISON OF THE RESULTS PROVIDED BY THE PROCEDURES FOR DEFLECTION COMPUTATION DESCRIBED ABOVE

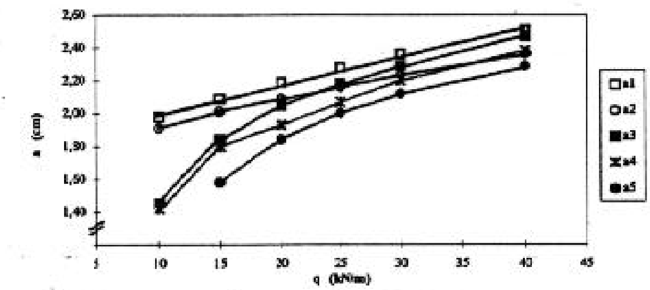
Deflection computation was performed by the above described methods at six load levels (10 to 40 kN/m uniformly distributed load). The cross section of the reinforced concrete beam examined was 30 by 50 cm, the length was 8m. C 25 concrete and S400 steel was assumed. The obtained midspan deflections are summarised below (see Tab.1).

The results obtained by the method described in subchapter 4.1. are represented by points a1 (assuming I_{II}^{+} constant, State II) and a3 (interpolation between States I and II by ζ).

The results obtained by the method described in subchapter 4.2. are not included in the graph, as this method overestimates the deflection because of the rotation at the fixed ends. (At q = 15 kN/m a = 2.32 cm)

The results obtained by the method described in subchapter 4.3. are represented by points a6 in the table, but they are not included in the graph either, as they are far greater than the other values.
The results obtained by the method described in subchapter 4.4. are represented by points a2 (assuming that the curvature calculated with I_{II}^{+} is constant) and a5 (interpolation between the curvatures in States I and II by ζ).
The results obtained by the finite element method are represented by points a4 – these values are regarded as the real midspan deflections of the beam.

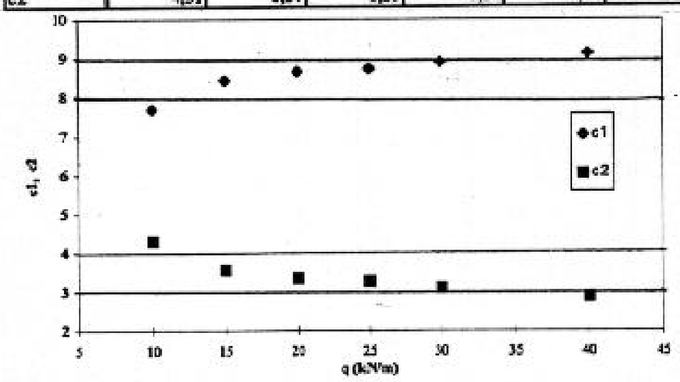
q (kN/m)	10	15	20	25	30	40	
a1	1,98	2,09	2,19	2,28	2,36	2,51	(4.1.)
a2	1,91	2,01	2,09	2,16	2,23	2,36	(4.4.)
a3	1,45	1,84	2,06	2,18	2,29	2,47	(4.1.)
a4	1,42	1,80	1,93	2,07	2,20	2,38	(4.5.)
a5	0,99	1,58	1,84	2,00	2,12	2,29	(4.4.)
a6	2,57	2,70	2,82	2,92	3,02	3,19	(4.3.)



Tab.1 Midspan deflections obtained by different methods, at six load levels

It can be seen in the graph that a3 values approach a1 values, and also that a5 values approach a2 values as the load increases. a(ζ) values approach a_{II} values as the ratio of the cracked segment of the beam to the total length increases, and thus there are shorter and shorter segments in State I (where the moment of inertia is I_I) that reduce the deflection.
The deflection values obtained from the finite element method (a4) are between the values provided by the method assuming constant curvature (4.4., a5) and the method assuming constant moment of inertia (4.1., a3).

q (kN/m)	10	15	20	25	30	40
c1	7,69	8,43	8,65	8,72	8,90	9,12
c2	4,31	3,57	3,35	3,28	3,10	2,88



Tab.2 Max. positive and negative moments by different methods at some load levels

Three of the above methods provide different moments at certain load levels (see Tab.2). According to the method assuming constant moment of inertia (4.1.),

$M^-_{max} = q\, l^2 / 12 = 8 * q\, l^2 / 96$ and $M^+_{max} = q\, l^2 / 24 = 4 * q\, l^2 / 96$

According to the method assuming constant curvature (4.4.),

$M^-_{max} = 3\, q\, l^2 / 32 = 9 * q\, l^2 / 96$ and $M^+_{max} = q\, l^2 / 32 = 3 * q\, l^2 / 96$

The finite element method, after performing some cycles of iteration, provides values in between the ones given by the other two methods. (see Tab.3)

$M^-_{max} = c1 * q\, l^2 / 96$ and $M^+_{max} = c2 * q\, l^2 / 96$

where c1 + c2 = 12 always, since the total moment is $12\, q\, l^2 / 96 = q\, l^2 / 8$.

6. CONCLUSION

By comparing the results of the methods described above, the conclusion can be drawn that either the method assuming constant moment of inertia (4.1.) or the one assuming constant curvature (4.4.) should be applied for deflection computation, as they provide deflection values that are good approximations of the deflection values regarded as real deflections. The reason for their accuracy is that they take corresponding moments and moments of inertia into account, and also, they provide zero rotation at the fixed ends.

The aim of the research is to find the correct moment diagrams of the column and middle strips of a flat slab, and also to work out a method for the computation of deflections of the slab, in which case the curves of the a-q diagram will be farther from each other.
Further results of the research will be presented at the Symposium.

7. LIST OF NOTATIONS

a, a_I, a_{II} deflection; deflection in States I and II
c1, c2 multiplicators
E_c modulus of elasticity of concrete
$I_{crk} = I_{II}$ moment of inertia of cracked, transformed section (tension stiffening ignored)
I_e effective moment of inertia ($I_{cr} < I_e < I_g$)
I_g moment of inertia of gross concrete section, neglecting reinforcement
I_I moment of inertia of uncracked, transformed section (State I)
l length of member
M_{crk} cracking moment
M^+_{max} maximum positive moment
q uniformly distributed load
ß coefficient in equation N.2, ß = 0.5
ζ interpolation coefficient
ψ curvature

8. REFERENCES

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